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Optimized Transfer Matrix Approach for Global Buckling Analysis: Bypassing Zero Matrix Inversion

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Abstract

The transfer matrix method has two main disadvantages concerning other numerical methods: numerical instability in extreme cases and the need to calculate the inverse of the zero matrix. This paper attempts to solve the second difficulty of the transfer matrix method, widely used for the global buckling analysis of beams in all engineering fields. In particular, the transfer matrix method necessarily requires the calculation of the inverse of the zero matrix to derive the element transfer matrix, resulting in high computational costs as the number of discretizations and the size of the matrix increases. To mitigate this challenge, this paper presents a transfer matrix method that directly computes the transfer matrix without requiring the inverse of the zero matrix. The method adopts a Laplacian approach, which involves the application of Laplace transforms to the equilibrium equations and subsequent inverse Laplace transforms to express displacements and internal forces relative to the zero point of the coordinate origin significantly reducing computational costs to a minimum. Numerical applications corroborate the effectiveness and superiority of the proposed approach.

Keywords

beam, calculations time, global buckling analysis, modified transfer matrix method, Laplace transform

1 Introduction

Buckling analysis in structures represented as continuous beams, with both uniform and variable properties, remains a significant focus in current research and has been extensively studied in mechanical, civil, industrial engineering, aerospace industry, and other engineering fields. In general, the equilibrium equations for the buckling analysis of structures lead to differential equations with variable coefficients. While closed-form analytical solutions are available for very simple uniform beams and specific particular cases, the prevalence of variable geometric, structural, and material properties in real beams requires advanced methodologies. Therefore, for solving the buckling problem, the literature prefers numerical methods. Numerous numerical methods have been employed so far to address the buckling issues in structures, including the finite strip method [1–3], generalized beam theory [4, 5], finite difference method [6], effective stiffness method [7], transfer matrix method [8, 9], various orders of Runge-Kutta method [10], differential transformation method [11], finite element method [12–15], differential quadrature method [16], matrix stiffness method [17], transfer matrix method, among others.

Of all existing numerical methods, this paper focuses on studying the transfer matrix method. The transfer matrix method was proposed in Germany during the 1950s and was extensively developed in the 1960s [18]. Along with other matrix methods, its development was influenced by the rapid advancement of computers after the Second World War. Essentially, it provides displacements and internal forces along the length of the element based on the initial displacements and internal forces through a sequence of transfer matrix multiplications of individual elements to form the global system.

It has been widely used in various engineering problems due to the following advantages over other numerical methods:

- The quality of results is independent of the number of subdivisions.

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- Exact results can be obtained for uniform beams by considering the structure as a single element, eliminating the need for spatial discretization.
- It involves a few degrees of freedom, and the order of the global transfer matrix is equal to the order of the differential equation determining its behavior. This order remains the same regardless of the number of discretizations of the structure (4 × 4 for Euler-Bernoulli beams, 4 × 4 for Timoshenko beams, 6 × 6 for sandwich beams, etc.).
- Features easy programming with minimal computational memory requirements.
- It is possible to obtain catalogs of typical beam transfer matrices for direct use without the need for rigorous mathematical deduction.
- The global transfer matrix is obtained directly by multiplying the transfer matrix of each discretized element (without increasing the size of the matrix).

However, two main disadvantages have limited its use in past decades, and these are currently being extensively addressed and investigated in the current literature:

- Under certain extreme conditions of numerical operations, results may experience non-physical numerical instabilities directly derived from rounding errors due to truncation when working with a finite number of digits (inevitable in all numerical methods). This problem has been partially addressed using a Riccati transformation [19].
- The need to calculate the inverse of the zero matrix (see the definition in Section 2, after Eq. (1)) to obtain the element transfer matrix.

This paper aims to address the second difficulty simply and efficiently by eliminating the need for calculating the inverse of the zero matrix, obtaining the element transfer matrix directly through a Laplacian approach applied to the global elastic buckling analysis of beams with uniform and variable properties.

Recently, different applications related to the analysis of global elastic buckling have been solved using the transfer matrix method. Considering the effect of rotational inertia and shear deformation, Wu and Chang [20] calculated the axial load effect on the free vibration of stepped beams with arbitrarily concentrated elements at their nodes. Lellep and Kraav [21] studied the influence of a surface crack in the incoming corners of steps on the stability of stepped beams. An interesting and promising application to the global stability analysis of masonry walls considering material nonlinearity was proposed by Bakeer [22]. An aeroelastic stability analysis of a high aspect ratio wing was developed by Duan and Zhang [23]. Demirkan and Artan [24] investigated the buckling of nanobeams based on a higher-order non-local Timoshenko beam model. The buckling of composite stepped beams influenced by slip at the interface of load-carrying layers was studied by Huang et al. [25]. Zheng et al. [26] studied the buckling of piles with partial support in an elastic medium of constant-coefficient layers. Xia [27] proposed a generalized foundation with three parameters (Winkler base, shear layer, and horizontal interface friction). The study of orthodontics in mandibular bone calculation and optimization was conducted by Suciu [28]. The stability of the tower column in an inclined cable-stayed bridge without stays and with a single tower float was studied by Kang et al. [29]. Avetisyan et al. [30] studied the stability of multiple-span beams of an infinite periodic structure. Jing et al. [31], through the Dirac function, studied the influence of a restraint at the midpoint of a compressed continuous beam. Li et al. [32] studied the effect of axial forces on multi-span beams resting on an elastic base.

The analysis of global elastic buckling can also be resolved by applying the transfer matrix method. Bozdogan [33], Bozdogan and Ozturk [34, 35], and Bozdogan [36], using a sandwich-type beam, extended the application to wall-frame buildings, asymmetric-plan shear wall and core buildings, and asymmetric shear wall structures. Recently, Pinto [37] studied the elastic buckling analysis of buildings and proposed a new generalized continuous sandwich beam that introduces a new kinematic rotation field associated with local shear stiffness. Because the transfer matrix resulted in a 6 × 6 order, he used the approach of Feyzollahzadeh and Bamdad [38] to reduce the order of the matrix to 3 × 3. In a later work, Pinto [39], using the continuum model proposed in [37] and introducing an approach similar to the one proposed in this paper to directly derive the transfer matrix, derived analytical and numerical solutions to study the static analysis of tall buildings. Additionally, the classical application of the continuum method to the analysis of global elastic buckling of buildings can also be solved using the transfer matrix method [40–45]. Finally, the latest applications to the buckling of coupled shear walls with uniform properties using the continuum method [46, 47] can be extended to study the analysis of coupled shear walls with non-uniform properties considering material nonlinearity.

The structured outline of this research is as follows: Section 2 provides a general overview of the classical transfer matrix method, establishing crucial fundamental concepts. Section 3 introduces an algorithm designed for efficient application, streamlining the computational process for structural engineers. This algorithm, serving as a practical bridge between theory and implementation, enhances accessibility. Section 4 demonstrates the versatility of the modified transfer matrix method (MTMM) through the global buckling analysis of classical beams, including those with fixed-base support. Attention is given to the Timoshenko beam and the sandwich-type beam. Using the sandwich-type continuous model, Section 5 show cases the numerical application of the global stability analysis of a three-bay coupled shear wall structure. To study a wide range of behavioral types, the number of levels has been varied from 5 to 30. Finally, Section 6 provides concluding remarks, summarizes the research findings, and highlights the significance and implications of the proposed approach.

2 Transfer matrix method (TMM)

The classical transfer matrix method (TMM) is a widely adopted technique for beam stability analysis. Its global acceptance is attributed to computational advantages over more complex methods such as one-dimensional finite element analysis. The TMM is particularly advantageous as it produces a transfer matrix of constant size, which matches the order of the governing differential equation, regardless of the discretization of the beam into elements. This results in a solution that requires only a few parameters, improving computational efficiency.

Because the coefficients of equilibrium equations in the form of differential equations are generally constant, the transfer matrix method has been widely used for static and dynamic analysis. The true strength of the method lies in the stability analysis, where the coefficients of the equilibrium equations are variable and allow multiple iterations with excellent convergence to eigenvalue solutions. However, a notable drawback is the need to compute the inverse of the zero matrix, which results in higher computational costs as the size of the matrix and the number of elements increase. This research seeks to address this limitation by eliminating the need to compute the inverse of the zero matrix and further improve the efficiency of the method.

The state vector $\boldsymbol{Z}_i(\boldsymbol{x}_i)$ contains the displacements and internal forces of the i-th element and is expressed as:

$$\boldsymbol{Z}_i(\boldsymbol{x}_i) = \boldsymbol{K}_i(\boldsymbol{x}_i)\boldsymbol{C}, \tag{1}$$

where $\boldsymbol{K}_i(\boldsymbol{x}_i)$ is a transfer matrix function and $\boldsymbol{C}$ is a vector of constants.

The transfer matrix function $\boldsymbol{K}_i(\boldsymbol{0})$ is referred to as the zero matrix and is obtained by evaluating $\boldsymbol{K}_i(\boldsymbol{x}_i)$ at the point $x_i = 0$, corresponding to the base of the i-th element.

The vector of constants $\boldsymbol{C}$ can be obtained by evaluating the state vector $\boldsymbol{Z}_i(\boldsymbol{x}_i)$ at the point $x_i = 0$.

$$\boldsymbol{C} = \boldsymbol{K}_i^{-1}(\boldsymbol{0})\boldsymbol{Z}_i(\boldsymbol{0}) \tag{2}$$

Introducing Eq. (2) into Eq. (1), the state vector $\boldsymbol{Z}_i(\boldsymbol{x}_i)$ is expressed as:

$$\boldsymbol{Z}_i(\boldsymbol{x}_i) = \boldsymbol{K}_i(\boldsymbol{x}_i)\boldsymbol{K}_i^{-1}(\boldsymbol{0})\boldsymbol{Z}_i(\boldsymbol{0}) = \boldsymbol{T}_i(\boldsymbol{x}_i)\boldsymbol{Z}_i(\boldsymbol{0}), \tag{3}$$

i.e.,

$$\boldsymbol{T}_i(\boldsymbol{x}_i) = \boldsymbol{K}_i(\boldsymbol{x}_i)\boldsymbol{K}_i^{-1}(\boldsymbol{0}), \tag{4}$$

where $\boldsymbol{T}_i(\boldsymbol{x}_i)$ is the transfer matrix relating displacements and internal forces between point x_i and the base of the i-th element.

It is noted that to obtain the transfer matrix $\boldsymbol{T}_i(\boldsymbol{x}_i)$, it is necessary to calculate the inverse of the zero matrix and then multiply it by $\boldsymbol{K}_i(\boldsymbol{x}_i)$.

An alternative approach to circumvent the calculation of the inverse of the zero matrix involves rearranging the coupled differential equations to determine the inverse of the zero matrix matrix with off-diagonal terms set to zero. This concept has been explored by Feyzollahzadeh and Bamdad [38]. While diagonalizing the inverse of the zero matrix proves advantageous in halving the size of the zero submatrices, this method is particularly beneficial for smaller matrix sizes, such as 2 × 2, 4 × 4, or even 6 × 6 matrices. However, as the size of the transfer matrix increases, analytical calculation of the inverse of the zero submatrices of the diagonal becomes challenging.

This limitation becomes evident in cases of extensive matrix sizes, as observed in the three-dimensional analysis of buildings where the transfer matrix size reaches 12 × 12 [36]. In such instances, the approach of diagonalizing the inverse of the zero matrix may encounter difficulties, prompting the need for alternative strategies to enhance computational feasibility.

This research modifies the derivation of the transfer matrix. An alternative method is presented, involving the application of the Laplace transform and the inverse Laplace transform to the coupled differential equations. This direct application of Laplace transforms enables the determination of the transfer matrix without the need to calculate the inverse of the zero matrix or the inverse of the submatrices of the diagonal, as suggested by

Feyzollahzadeh and Bamdad [38]. The proposed approach offers an optimized solution that improves computational efficiency compared to traditional methods.

3 Procedure of computation

In this section the proposed algorithm for the MTMM is described, which facilitates the direct derivation of the transfer matrix without the requirement to compute the inverse of the zero matrix. The steps to implement the algorithm are as follows:

- Determine the Lagrangian functional that contains the potential energy and the work done by the external force.
- Apply Hamilton's principle to the Lagrangian functional and derive the equilibrium equations, constitutive laws, and boundary conditions.
- Apply the Laplace transform to the uncoupled and coupled equations of equilibrium.
- Solve the Laplace transforms of displacements in terms of displacements and internal forces of the zero point of the coordinate origin.
- Apply the inverse Laplace transform to the decoupled Laplace transforms of the displacements to find the functions of the displacements in terms of the position variable *x*.
- Derive the internal forces as bending moments and shear forces in terms of the position variable *x*.
- Write the expressions for displacement and internal forces in matrix form to obtain the transfer matrix of the *i*-th element.
- Calculate the transfer matrix and force matrix of the entire beam.
- The internal forces at the zero point must be calculated based on the boundary conditions of the beam.

To obtain non-trivial solutions, the determinant of the matrix accompanying the internal forces matrix must be zero.

4 Modified transfer matrix method (MTMM)

4.1 Case study: Timoshenko beam

This section derives the transfer matrix for the elastic global buckling analysis of the Timoshenko beam. The Timoshenko beam results from the series coupling of the bending beam and the shear beam, it has two characteristic stiffnesses associated with a bending stiffness (K_b) and a shear stiffness (K_s). In addition, it has two kinematic fields that describe lateral (u) and rotational (θ) displacement.

The total potential energy of the beam (Fig. 1) is expressed as:

$$\Pi = \frac{1}{2}\int_0^L \left\{K_b\theta'^2(x) + K_s\left[u'(x)-\theta(x)\right]^2\right\}dx - \frac{1}{2}\int_0^L Pu'^2(x)\,dx + \frac{1}{2}K_{T1}u^2(0) + \frac{1}{2}K_{R1}\theta^2(0) + \frac{1}{2}K_{T2}u^2(L) + \frac{1}{2}K_{R2}\theta^2(L), \quad (5)$$

where $\frac{1}{2}\int_0^L K_b\theta'^2(x)\,dx$, $\frac{1}{2}\int_0^L K_s\left[u'(x)-\theta(x)\right]'^2 dx$, $\frac{1}{2}\int_0^L Pu'^2(x)\,dx$ correspond to the potential energy in bending, shear and the work done by the vertical force respectively. The energies associated with the elastic springs are also shown.

The Lagrangian functional is solved by applying Hamilton's principle:

$$\delta\int_{t_1}^{t_2} \Pi\,dt = 0, \quad (6)$$

i.e.,

$$\begin{aligned}\delta\Pi = &\int_0^H \left[-(K_s-P)u''(x) + K_s\theta'(x)\right]\delta u(x)\,dx + \int_0^H \left[-K_b\theta''(x) - K_s\left[u'(x)-\theta(x)\right]\right]\delta\theta(x)\,dx + \left[K_b\theta'(x)\right]\delta\theta(x)\Big|_0^H \\ &+\left[(K_s-P)u'(x) - K_s\theta(x)\right]\delta u(x)\Big|_0^H + K_{T1}u(0)\delta u(0) + K_{R1}\theta(0)\delta\theta(0) + K_{T2}u(L)\delta u(L) + K_{R2}\theta(L)\delta\theta(L) = 0.\end{aligned} \quad (7)$$

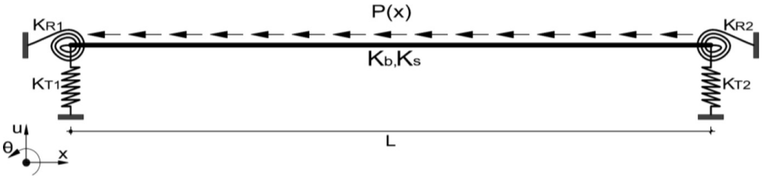

Fig. 1 Definition of the geometry and coordinate system of the Timoshenko-type beam

The parameters β and λ_1 are defined as:

$$\beta = \sqrt{\frac{K_s}{K_b}}, \tag{8}$$

$$\lambda_1 = \sqrt{\frac{PK_s}{K_b\left(K_s - P\right)}}. \tag{9}$$

The internal forces can be written as:

$$M(x) = K_b\theta'(x), \tag{10}$$

$$V(x) = \left(K_s - P\right)u'(x) - K_s\theta(x), \tag{11}$$

or:

$$M^*(x) = \frac{M(x)}{K_b} = \theta'(x), \tag{12}$$

$$V^*(x) = \frac{V(x)}{K_s - P} = u'(x) - \left(\frac{\lambda_1^2 + \beta^2}{\beta^2}\right)\theta(x). \tag{13}$$

Derived from achieving stationarity, two coupled equilibrium equations are obtained:

$$-\left(K_s - P\right)u''(x) + K_s\theta'(x) = 0, \tag{14}$$

$$-K_b\theta''(x) - K_s\left[u'(x) - \theta(x)\right] = 0. \tag{15}$$

By substituting the parameters of Eqs. (8) and (9) into Eqs. (14) and (15), the coupled differential equations can be expressed as:

$$-u''(x) + \left(\frac{\lambda_1^2 + \beta^2}{\beta^2}\right)\theta'(x) = 0, \tag{16}$$

$$-\theta''(x) - \beta^2 u'(x) + \beta^2\theta(x) = 0. \tag{17}$$

Let u and θ be a function defined for $x \geq 0$. Then it is said that the integrals:

$$U(s) = L\left[u(x)\right] = \int_0^\infty u(x)e^{-sx}dx, \tag{18}$$

$$\Phi(s) = L\left[\theta(x)\right] = \int_0^\infty \theta(x)e^{-sx}dx, \tag{19}$$

are the Laplace transform of the lateral and rotational displacement, respectively.

Applying Laplace transforms to coupled differential equations:

$$\begin{bmatrix} -s^2 & \left(\frac{\lambda_1^2 + \beta^2}{\beta^2}\right)s \\ -\beta^2 s & -\left(s^2 - \beta^2\right) \end{bmatrix} \begin{Bmatrix} U(s) \\ \Phi(s) \end{Bmatrix} = \begin{Bmatrix} -su(0) - V^*(0) \\ -\beta^2 u(0) - s\theta(0) - M^*(0) \end{Bmatrix}, \tag{20}$$

i.e.,

$$U(s) = \left(\frac{1}{s}\right)u(0) + \left(\frac{\lambda_1^2 + \beta^2}{\beta^2}\right)\left(\frac{1}{s^2 + \lambda_1^2}\right)\theta(0) + \left(\frac{\lambda_1^2 + \beta^2}{\beta^2}\right)\left[\frac{1}{s\left(s^2 + \lambda_1^2\right)}\right]M^*(0) + \left[\frac{s^2 - \beta^2}{s^2\left(s^2 + \lambda_1^2\right)}\right]V^*(0), \tag{21}$$

$$\Phi(s) = \left(\frac{s}{s^2 + \lambda_1^2}\right)\theta(0) + \left(\frac{1}{s^2 + \lambda_1^2}\right)M^*(0) - \left[\frac{\beta^2}{s\left(s^2 + \lambda_1^2\right)}\right]V^*(0). \tag{22}$$

The displacements are obtained by solving the Laplace transform of the coupled differential equations and then applying the inverse Laplace transform.

$$u(x) = L^{-1}\left(\frac{1}{s}\right)u(0) + \left(\frac{\lambda_1^2 + \beta^2}{\beta^2}\right)L^{-1}\left(\frac{1}{s^2 + \lambda_1^2}\right)\theta(0) + \left(\frac{\lambda_1^2 + \beta^2}{\beta^2}\right)L^{-1}\left[\frac{1}{s\left(s^2 + \lambda_1^2\right)}\right]M^*(0) + L^{-1}\left[\frac{1}{s^2 + \lambda_1^2} - \frac{\beta^2}{s^2\left(s^2 + \lambda_1^2\right)}\right]V^*(0) \tag{23}$$

$$\theta(x) = L^{-1}\left(\frac{s}{s^2 + \lambda_1^2}\right)\theta(0) + L^{-1}\left(\frac{1}{s^2 + \lambda_1^2}\right)M^*(0) - L^{-1}\left[\frac{\beta^2}{s\left(s^2 + \lambda_1^2\right)}\right]V^*(0) \tag{24}$$

After some simple algebraic manipulations, the inverses of the Laplace transforms are expressed as:

$$L^{-1}\left(\frac{1}{s}\right)=1, \tag{25}$$

$$L^{-1}\left(\frac{s}{s^2+\lambda_1^2}\right)=\cos(\lambda_1 x), \tag{26}$$

$$L^{-1}\left(\frac{1}{s^2+\lambda_1^2}\right)=\frac{\sin(\lambda_1 x)}{\lambda_1}, \tag{27}$$

$$L^{-1}\left[\frac{1}{s(s^2+\lambda_1^2)}\right]=\frac{1-\cos(\lambda_1 x)}{\lambda_1^2}, \tag{28}$$

$$L^{-1}\left[\frac{1}{s^2(s^2+\lambda_1^2)}\right]=\frac{\lambda_1 x-\sin(\lambda_1 x)}{\lambda_1^3}. \tag{29}$$

The above terms are typically sufficient for writing the Laplace and inverse Laplace transforms for beams in a closed analytic form.

Displacements are obtained by introducing Eqs. (25) to (29) into Eqs. (23) and (24), i.e.,

$$u(x)=u(0)+\left(\frac{\lambda_1^2+\beta^2}{\beta^2}\right)\left[\frac{\sin(\lambda_1 x)}{\lambda_1}\right]\theta(0)+\left(\frac{\lambda_1^2+\beta^2}{\beta^2}\right)\left[\frac{1-\cos(\lambda_1 x)}{\lambda_1^2}\right]M^*(0)-\left[\frac{\beta^2\lambda_1 x-(\lambda_1^2+\beta^2)\sin(\lambda_1 x)}{\lambda_1^3}\right]V^*(0), \tag{30}$$

$$\theta(x)=\cos(\lambda_1 x)\theta(0)+\left[\frac{\sin(\lambda_1 x)}{\lambda_1}\right]M^*(0)-\beta^2\left[\frac{1-\cos(\lambda_1 x)}{\lambda_1^2}\right]V^*(0). \tag{31}$$

The internal forces are expressed as:

$$M^*(x)=\theta'(x)=-\lambda_1\sin(\lambda_1 x)\theta(0)+\cos(\lambda_1 x)M^*(0)-\beta^2\left[\frac{\sin(\lambda_1 x)}{\lambda_1}\right]V^*(0), \tag{32}$$

$$V(x)=u'(x)-\left(\frac{\lambda_1^2+\beta^2}{\beta^2}\right)\theta(x)=V^*(0). \tag{33}$$

Writing in matrix form:

$$\begin{Bmatrix} u(x)\\ \theta(x)\\ M^*(x)\\ V^*(x)\end{Bmatrix}=\begin{bmatrix} 1 & \left(\frac{\lambda_1^2+\beta^2}{\beta^2}\right)\left[\frac{\sin(\lambda_1 x)}{\lambda_1}\right] & \left(\frac{\lambda_1^2+\beta^2}{\beta^2}\right)\left[\frac{1-\cos(\lambda_1 x)}{\lambda_1^2}\right] & -\left[\frac{\beta^2\lambda_1 x-(\lambda_1^2+\beta^2)\sin(\lambda_1 x)}{\lambda_1^3}\right]\\ 0 & \cos(\lambda_1 x) & \frac{\sin(\lambda_1 x)}{\lambda_1} & -\beta^2\left[\frac{1-\cos(\lambda_1 x)}{\lambda_1^2}\right]\\ 0 & -\lambda_1\sin(\lambda_1 x) & \cos(\lambda_1 x) & -\beta^2\left[\frac{\sin(\lambda_1 x)}{\lambda_1}\right]\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{Bmatrix} u(0)\\ \theta(0)\\ M^*(0)\\ V^*(0)\end{Bmatrix}. \tag{34}$$

The transfer matrix is:

$$\boldsymbol{T}(\boldsymbol{x})=\begin{bmatrix} 1 & \left(\frac{\lambda_1^2+\beta^2}{\beta^2}\right)\left[\frac{\sin(\lambda_1 x)}{\lambda_1}\right] & \left(\frac{\lambda_1^2+\beta^2}{\beta^2}\right)\left[\frac{1-\cos(\lambda_1 x)}{\lambda_1^2}\right] & -\left[\frac{\beta^2\lambda_1 x-(\lambda_1^2+\beta^2)\sin(\lambda_1 x)}{\lambda_1^3}\right]\\ 0 & \cos(\lambda_1 x) & \frac{\sin(\lambda_1 x)}{\lambda_1} & -\beta^2\left[\frac{1-\cos(\lambda_1 x)}{\lambda_1^2}\right]\\ 0 & -\lambda_1\sin(\lambda_1 x) & \cos(\lambda_1 x) & -\beta^2\left[\frac{\sin(\lambda_1 x)}{\lambda_1}\right]\\ 0 & 0 & 0 & 1\end{bmatrix}. \tag{35}$$

For the continuous beam divided into n elements, the relationship between the displacements and the internal forces at the final and initial node of the continuous model is obtained by successively applying the multiplication of transfer matrices [22], i.e.,

$$\begin{Bmatrix} u(h_n) \\ \theta(h_n) \\ M^*(h_n) \\ V^*(h_n) \end{Bmatrix} = \boldsymbol{T_n}(\boldsymbol{h_n})\boldsymbol{T_{n-1}}(\boldsymbol{h_{n-1}})\ldots\boldsymbol{T_3}(\boldsymbol{h_3})\boldsymbol{T_2}(\boldsymbol{h_2})\boldsymbol{T_1}(\boldsymbol{h_1})\begin{Bmatrix} u(0) \\ \theta(0) \\ M^*(0) \\ V^*(0) \end{Bmatrix} = \boldsymbol{T}\begin{Bmatrix} u(0) \\ \theta(0) \\ M^*(0) \\ V^*(0) \end{Bmatrix}, \tag{36}$$

where [22]:

$$\boldsymbol{T} = \boldsymbol{T_n}(\boldsymbol{h_n})\boldsymbol{T_{n-1}}(\boldsymbol{h_{n-1}})\ldots\boldsymbol{T_3}(\boldsymbol{h_3})\boldsymbol{T_2}(\boldsymbol{h_2})\boldsymbol{T_1}(\boldsymbol{h_1}) = \prod_{k=n}^{1}\boldsymbol{T_k}(\boldsymbol{h_k}). \tag{37}$$

4.2 Case study: Sandwich beam

Section 4.2 derives the transfer matrix for the global elastic buckling analysis of the sandwich beam. The sandwich beam results from the parallel coupling of the Timoshenko beam (it couples the bending beam and the shear beam in series) and the bending beam. It has three characteristic stiffnesses associated with global bending (K_{b1}), global shear (K_{s1}), and local bending (K_{b2}). In addition, it has two kinematic fields that describe lateral (u) and rotational (θ) displacement.

The total potential energy of the beam (Fig. 2) is expressed as:

$$\Pi = \frac{1}{2}\int_0^L\left\{K_{b1}\theta'^2(x) + K_{s1}\left[u'(x)-\theta(x)\right]^2 + K_{b2}u''^2(x)\right\}dx - \frac{1}{2}\int_0^L Pu'^2(x)\,dx + \frac{1}{2}K_{T1}u^2(0) + \frac{1}{2}K_{R1}\theta^2(0) + \frac{1}{2}K_{T2}u^2(L) + \frac{1}{2}K_{R2}\theta^2(L), \tag{38}$$

where $\frac{1}{2}\int_0^L K_{b1}\theta'^2(x)dx$, $\frac{1}{2}\int_0^L K_{s1}\left[u'(x)-\theta(x)\right]^2(x)dx$, $\frac{1}{2}\int_0^L K_{b2}u''^2(x)dx$, $\frac{1}{2}\int_0^L Pu'^2(x)$ correspond to the potential energy in global bending, global shear, local bending and the work done by the vertical force respectively. Energies associated with elastic springs are also shown.

The Lagrangian functional is solved by applying Hamilton's principle:

$$\delta\int_{t_1}^{t_2}\Pi dt = 0, \tag{39}$$

i.e.,

$$\begin{aligned}\delta\Pi = &\int_0^H\left[K_{b2}u''''(x) - (K_{s1}-P)u''(x) + K_{s1}\theta'(x)\right]\delta u(x)dx + \int_0^H\left[-K_{b1}\theta''(x) - K_{s1}\left[u'(x)-\theta(x)\right]\right]\delta\theta(x)dx \\ &+\left[K_{b1}\theta'(x)\right]\delta\theta(x)\Big|_0^H + \left[K_{b2}u''(x)\right]\delta u'(x)\Big|_0^H + \left[-K_{b2}u'''(x) + (K_{s1}-P)u'(x) - K_{s1}\theta(x)\right]\delta u(x)\Big|_0^H \\ &+K_{T1}u(0)\delta u(0) + K_{R1}\theta(0)\delta\theta(0) + K_{T2}u(L)\delta u(L) + K_{R2}\theta(L)\delta\theta(L) = 0.\end{aligned} \tag{40}$$

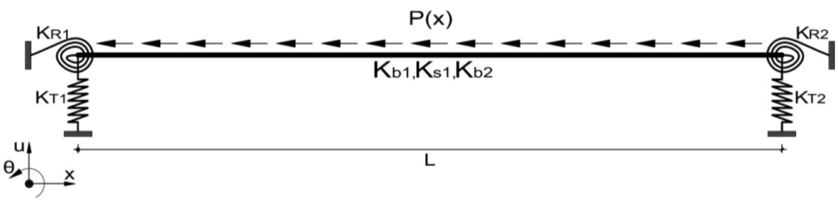

Fig. 2 Definition of the geometry and coordinate system of the sandwich beam

The parameters α, κ and λ_2 are defined as:

$$\alpha = \sqrt{\frac{K_{s1}}{K_{b2}}}, \tag{41}$$

$$\kappa = \sqrt{1 + \frac{K_{b2}}{K_{b1}}}, \tag{42}$$

$$\lambda_2 = \sqrt{\frac{P}{K_{b2}}}. \tag{43}$$

The internal forces can be written as:

$$M_1(x) = K_{b1}\theta'(x), \tag{44}$$

$$M_2(x) = K_{b2}u''(x), \tag{45}$$

$$V(x) = -K_{b2}u'''(x) + (K_{s1} - P)u'(x) - K_{s1}\theta(x), \tag{46}$$

or:

$$M_1^*(x) = \frac{M_1(x)}{K_{b1}} = \theta'(x), \tag{47}$$

$$M_2^*(x) = \frac{M_2(x)}{K_{b2}} = u''(x), \tag{48}$$

$$V^*(x) = \frac{V(x)}{K_{b2}} = -u'''(x) + (\alpha^2 - \lambda_2^2)u'(x) - \alpha^2\theta(x). \tag{49}$$

Derived from achieving stationarity, two coupled equilibrium equations are obtained:

$$K_{b2}u''''(x) - (K_{s1} - P)u''(x) + K_{s1}\theta'(x) = 0, \tag{50}$$

$$-K_{b1}\theta''(x) - K_{s1}\left[u'(x) - \theta(x)\right]. \tag{51}$$

By substituting the parameters of Eqs. (41) to (43) into Eqs. (50) and (51), the coupled differential equations can be expressed as:

$$u''''(x) - (\alpha^2 - \lambda_2^2)u''(x) + \alpha^2\theta'(x) = 0, \tag{52}$$

$$-\theta''(x) - \alpha^2(\kappa^2 - 1)u'(x) + \alpha^2(\kappa^2 - 1)\theta(x) = 0. \tag{53}$$

Let u and θ be a function defined for $x \geq 0$. Then it is said that the integrals:

$$U(s) = L\left[u(x)\right] = \int_0^\infty u(x)e^{-sx}dx, \tag{54}$$

$$\Phi(s) = L\left[\theta(x)\right] = \int_0^\infty L\theta(x)e^{-sx}dx, \tag{55}$$

are the Laplace transform of the lateral and rotational displacement, respectively.

Applying Laplace transforms to coupled differential equations:

$$\begin{bmatrix} s^2\left[s^2 - (\alpha^2 - \lambda_2^2)\right] & \alpha^2 s \\ -\alpha^2(\kappa^2 - 1)s & -\left[s^2 - \alpha^2(\kappa^2 - 1)\right] \end{bmatrix} \begin{Bmatrix} U(s) \\ \Phi(s) \end{Bmatrix} = \begin{Bmatrix} s^2\left[s^2 - (\alpha^2 - \lambda_2^2)\right]u(0) + s^2u'(0) + sM_2^*(0) - V^*(0) \\ -\alpha^2(\kappa^2 - 1)u(0) - s\theta(0) - M_1^*(0) \end{Bmatrix}, \tag{56}$$

i.e.,

$$U(s) = \left(\frac{1}{s}\right)u(0) + \left[\frac{s^2 - \alpha^2(\kappa^2 - 1)}{(s^2 - \xi_1^2)(s^2 + \xi_2^2)}\right]u'(0) - \left[\frac{\alpha^2}{(s^2 - \xi_1^2)(s^2 + \xi_2^2)}\right]\theta(0) - \left[\frac{\alpha^2}{s(s^2 - \xi_1^2)(s^2 + \xi_2^2)}\right]M_1^*(0) + \left[\frac{s^2 - \alpha^2(\kappa^2 - 1)}{s(s^2 - \xi_1^2)(s^2 + \xi_2^2)}\right]M_2^*(0) - \left[\frac{s^2 - \alpha^2(\kappa^2 - 1)}{s(s^2 - \xi_1^2)(s^2 + \xi_2^2)}\right]V^*(0), \tag{57}$$

$$\Phi(s) = -\left[\frac{\alpha^2(\kappa^2 - 1)s}{(s^2 - \xi_1^2)(s^2 + \xi_2^2)}\right]u'(0) + \left[\frac{s^3 - (\alpha^2 - \lambda_2^2)s}{(s^2 - \xi_1^2)(s^2 + \xi_2^2)}\right]\theta(0) + \left[\frac{s^2 - (\alpha^2 - \lambda_2^2)}{(s^2 - \xi_1^2)(s^2 + \xi_2^2)}\right]M_1^*(0) - \left[\frac{\alpha^2(\kappa^2 - 1)}{(s^2 - \xi_1^2)(s^2 + \xi_2^2)}\right]M_2^*(0) + \left[\frac{\alpha^2(\kappa^2 - 1)}{s(s^2 - \xi_1^2)(s^2 + \xi_2^2)}\right]V^*(0), \tag{58}$$

where:

$$\xi_1 = \sqrt{\frac{(\alpha^2\kappa^2 - \lambda_2^2) + \sqrt{(\alpha^2\kappa^2 - \lambda_2^2)^2 + 4\alpha^2(\kappa^2 - 1)\lambda_2^2}}{2}}, \tag{59}$$

$$\xi_2 = \sqrt{\frac{-(\alpha^2\kappa^2 - \lambda_2^2) + \sqrt{(\alpha^2\kappa^2 - \lambda_2^2)^2 + 4\alpha^2(\kappa^2 - 1)\lambda_2^2}}{2}}. \tag{60}$$

The displacements are obtained by solving the Laplace transform of the coupled differential equations and then applying the inverse Laplace transform.

$$
\begin{aligned}
u(x) &= L^{-1}\left(\frac{1}{s}\right)u(0) + L^{-1}\left[\frac{s^2}{(s^2-\xi_1^2)(s^2+\xi_2^2)} - \frac{\alpha^2(\kappa^2-1)}{(s^2-\xi_1^2)(s^2+\xi_2^2)}\right]u'(0) - L^{-1}\left[\frac{\alpha^2}{(s^2-\xi_1^2)(s^2+\xi_2^2)}\right]\theta(0) \\
&-L^{-1}\left[\frac{\alpha^2}{s(s^2-\xi_1^2)(s^2+\xi_2^2)}\right]M_1^*(0) + L^{-1}\left[\frac{s}{(s^2-\xi_1^2)(s^2+\xi_2^2)} - \frac{\alpha^2(\kappa^2-1)}{s(s^2-\xi_1^2)(s^2+\xi_2^2)}\right]M_2^*(0) \\
&-L^{-1}\left[\frac{1}{(s^2-\xi_1^2)(s^2+\xi_2^2)} - \frac{\alpha^2(\kappa^2-1)}{s^2(s^2-\xi_1^2)(s^2+\xi_2^2)}\right]V^*(0)
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
\theta(x) &= -L^{-1}\left[\frac{\alpha^2(\kappa^2-1)s}{(s^2-\xi_1^2)(s^2+\xi_2^2)}\right]u'(0) + L^{-1}\left[\frac{s^3}{(s^2-\xi_1^2)(s^2+\xi_2^2)} - \frac{(\alpha^2-\lambda_2^2)s}{(s^2-\xi_1^2)(s^2+\xi_2^2)}\right]\theta(0) \\
&+L^{-1}\left[\frac{s^2}{(s^2-\xi_1^2)(s^2+\xi_2^2)} - \frac{(\alpha^2-\lambda_2^2)}{(s^2-\xi_1^2)(s^2+\xi_2^2)}\right]M_1^*(0) - L^{-1}\left[\frac{\alpha^2(\kappa^2-1)}{(s^2-\xi_1^2)(s^2+\xi_2^2)}\right]M_2^*(0) \\
&+L^{-1}\left[\frac{\alpha^2(\kappa^2-1)}{s(s^2-\xi_1^2)(s^2+\xi_2^2)}\right]V^*(0)
\end{aligned}
\tag{62}
$$

After some simple algebraic manipulations, the inverses of the Laplace transforms are expressed as:

$$L^{-1}\left[\frac{s^3}{(s^2-\xi_1^2)(s^2+\xi_2^2)}\right] = \left(\frac{\xi_1^2}{\xi_1^2+\xi_2^2}\right)\cosh(\xi_1 x) + \left(\frac{\xi_2^2}{\xi_1^2+\xi_2^2}\right)\cos(\xi_2 x), \tag{63}$$

$$L^{-1}\left[\frac{s^2}{(s^2-\xi_1^2)(s^2+\xi_2^2)}\right] = \left(\frac{\xi_1}{\xi_1^2+\xi_2^2}\right)\sinh(\xi_1 x) + \left(\frac{\xi_2}{\xi_1^2+\xi_2^2}\right)\sin(\xi_2 x), \tag{64}$$

$$L^{-1}\left[\frac{s}{(s^2-\xi_1^2)(s^2+\xi_2^2)}\right] = \left(\frac{1}{\xi_1^2+\xi_2^2}\right)\cosh(\xi_1 x) - \left(\frac{1}{\xi_1^2+\xi_2^2}\right)\cos(\xi_2 x), \tag{65}$$

$$L^{-1}\left[\frac{1}{(s^2-\xi_1^2)(s^2+\xi_2^2)}\right] = \left(\frac{1}{\xi_1^2+\xi_2^2}\right)\frac{\sinh(\xi_1 x)}{\xi_1} - \left(\frac{1}{\xi_1^2+\xi_2^2}\right)\frac{\sin(\xi_2 x)}{\xi_2}, \tag{66}$$

$$L^{-1}\left[\frac{1}{s(s^2-\xi_1^2)(s^2+\xi_2^2)}\right] = -\left(\frac{1}{\xi_1^2\xi_2^2}\right) + \left[\frac{1}{\xi_1^2(\xi_1^2+\xi_2^2)}\right]\cosh(\xi_1 x) + \left[\frac{1}{\xi_2^2(\xi_1^2+\xi_2^2)}\right]\cos(\xi_2 x), \tag{67}$$

$$L^{-1}\left[\frac{1}{s^2(s^2-\xi_1^2)(s^2+\xi_2^2)}\right] = -\left(\frac{1}{\xi_1^2\xi_2^2}\right)x + \left[\frac{1}{\xi_1^2(\xi_1^2+\xi_2^2)}\right]\frac{\sinh(\xi_1 x)}{\xi_1} + \left[\frac{1}{\xi_2^2(\xi_1^2+\xi_2^2)}\right]\frac{\sin(\xi_2 x)}{\xi_2}. \tag{68}$$

The above terms are typically sufficient for writing the Laplace and inverse Laplace transforms for beams in a closed analytic form.

Displacements are obtained by introducing Eqs. (63) to (68) into Eqs. (61) and (62), i.e.,

$$\begin{aligned} u(x) &= u(0) + \left(\frac{1}{\xi_1^2+\xi_2^2}\right)\left\{\left[\xi_1^2-\alpha^2\left(\kappa^2-1\right)\right]\frac{\sinh(\xi_1 x)}{\xi_1}+\left[\xi_2^2+\alpha^2\left(\kappa^2-1\right)\right]\frac{\sin(\xi_2 x)}{\xi_2}\right\}u'(0) \\ &-\left(\frac{\alpha^2}{\xi_1^2+\xi_2^2}\right)\left[\frac{\sinh(\xi_1 x)}{\xi_1}-\frac{\sin(\xi_2 x)}{\xi_2}\right]\theta(0)-\alpha^2\left\{-\frac{1}{\xi_1^2\xi_2^2}+\left(\frac{1}{\xi_1^2+\xi_2^2}\right)\left[\frac{1}{\xi_1^2}\cosh(\xi_1 x)+\frac{1}{\xi_2^2}\cos(\xi_2 x)\right]\right\}M_1^*(0) \\ &+\left\{\frac{\alpha^2\left(\kappa^2-1\right)}{\xi_1^2\xi_2^2}+\left(\frac{1}{\xi_1^2+\xi_2^2}\right)\left[\frac{\xi_1^2-\alpha^2\left(\kappa^2-1\right)}{\xi_1^2}\right]\cosh(\xi_1 x)-\left(\frac{1}{\xi_1^2+\xi_2^2}\right)\left[\frac{\xi_2^2+\alpha^2\left(\kappa^2-1\right)}{\xi_2^2}\right]\cos(\xi_2 x)\right\}M_2^*(0) \\ &-\left\{\frac{\alpha^2\left(\kappa^2-1\right)}{\xi_1^2\xi_2^2}x+\left(\frac{1}{\xi_1^2+\xi_2^2}\right)\left[\frac{\xi_1^2-\alpha^2\left(\kappa^2-1\right)}{\xi_1^2}\right]\frac{\sinh(\xi_1 x)}{\xi_1}-\left(\frac{1}{\xi_1^2+\xi_2^2}\right)\left[\frac{\xi_2^2+\alpha^2\left(\kappa^2-1\right)}{\xi_2^2}\right]\frac{\sin(\xi_2 x)}{\xi_2}\right\}V^*(0), \end{aligned} \tag{69}$$

$$\begin{aligned} u'(x) &= \left(\frac{1}{\xi_1^2+\xi_2^2}\right)\left\{\left[\xi_1^2-\alpha^2\left(\kappa^2-1\right)\right]\cosh(\xi_1 x)+\left[\xi_2^2+\alpha^2\left(\kappa^2-1\right)\right]\cos(\xi_2 x)\right\}u'(0) \\ &-\left(\frac{\alpha^2}{\xi_1^2+\xi_2^2}\right)\left[\cosh(\xi_1 x)-\cos(\xi_2 x)\right]\theta(0)-\left(\frac{\alpha^2}{\xi_1^2+\xi_2^2}\right)\left[\frac{\sinh(\xi_1 x)}{\xi_1}-\frac{\sin(\xi_2 x)}{\xi_2}\right]M_1^*(0) \\ &+\left(\frac{1}{\xi_1^2+\xi_2^2}\right)\left\{\left[\xi_1^2-\alpha^2\left(\kappa^2-1\right)\right]\frac{\sinh(\xi_1 x)}{\xi_1}+\left[\xi_2^2+\alpha^2\left(\kappa^2-1\right)\right]\frac{\sin(\xi_2 x)}{\xi_2}\right\}M_2^*(0) \\ &-\left\{\frac{\alpha^2\left(\kappa^2-1\right)}{\xi_1^2\xi_2^2}+\left(\frac{1}{\xi_1^2+\xi_2^2}\right)\left[\frac{\xi_1^2-\alpha^2\left(\kappa^2-1\right)}{\xi_1^2}\right]\cosh(\xi_1 x)-\left(\frac{1}{\xi_1^2+\xi_2^2}\right)\left[\frac{\xi_2^2+\alpha^2\left(\kappa^2-1\right)}{\xi_2^2}\right]\cos(\xi_2 x)\right\}V^*(0), \end{aligned} \tag{70}$$

$$\begin{aligned} \theta(x) &= -\frac{\alpha^2\left(\kappa^2-1\right)}{\xi_1^2+\xi_2^2}\left[\cosh(\xi_1 x)-\cos(\xi_2 x)\right]u'(0) \\ &+\left(\frac{1}{\xi_1^2+\xi_2^2}\right)\left\{\left[\xi_1^2-\left(\alpha^2-\lambda_2^2\right)\right]\cosh(\xi_1 x)+\left[\xi_2^2+\left(\alpha^2-\lambda_2^2\right)\right]\cos(\xi_2 x)\right\}\theta(0) \\ &+\left(\frac{1}{\xi_1^2+\xi_2^2}\right)\left\{\left[\xi_1^2-\left(\alpha^2-\lambda_2^2\right)\right]\frac{\sinh(\xi_1 x)}{\xi_1}+\left[\xi_2^2+\left(\alpha^2-\lambda_2^2\right)\right]\frac{\sin(\xi_2 x)}{\xi_2}\right\}M_1^*(0) \\ &-\frac{\alpha^2\left(\kappa^2-1\right)}{\xi_1^2+\xi_2^2}\left[\frac{\sinh(\xi_1 x)}{\xi_1}-\frac{\sin(\xi_2 x)}{\xi_2}\right]M_2^*(0)+\alpha^2\left(\kappa^2-1\right)\left\{-\frac{1}{\xi_1^2\xi_2^2}+\frac{1}{\xi_1^2\left(\xi_1^2+\xi_2^2\right)}\cosh(\xi_1 x)+\frac{1}{\xi_2^2\left(\xi_1^2+\xi_2^2\right)}\cos(\xi_2 x)\right\}V^*(0), \end{aligned} \tag{71}$$

i.e.,

$$u(x)=u(0)+t_{12}(x)u'(0)+t_{13}(x)\theta(0)+t_{14}(x)M_1^*(0)+t_{15}(x)M_2^*(0)+t_{16}(x)V^*(0), \tag{72}$$

$$u'(x)=t_{22}(x)u'(0)+t_{23}(x)\theta(0)+t_{24}(x)M_1^*(0)+t_{25}(x)M_2^*(0)+t_{26}(x)V^*(0), \tag{73}$$

$$\theta(x)=t_{32}(x)u'(0)+t_{33}(x)\theta(0)+t_{34}(x)M_1^*(0)+t_{35}(x)M_2^*(0)+t_{36}(x)V^*(0). \tag{74}$$

The internal forces are expressed as:

$$\begin{aligned} M_1^*(x) &= \theta'(x) = -\frac{\alpha^2\left(\kappa^2-1\right)}{\xi_1^2+\xi_2^2}\left[\xi_1\sinh(\xi_1 x)+\xi_2\sin(\xi_2 x)\right]u'(0) \\ &+\left(\frac{1}{\xi_1^2+\xi_2^2}\right)\left\{\left[\xi_1^2-\left(\alpha^2-\lambda_2^2\right)\right]\xi_1\sinh(\xi_1 x)-\left[\xi_2^2+\left(\alpha^2-\lambda_2^2\right)\right]\xi_2\sin(\xi_2 x)\right\}\theta(0) \\ &+\left(\frac{1}{\xi_1^2+\xi_2^2}\right)\left\{\left[\xi_1^2-\left(\alpha^2-\lambda_2^2\right)\right]\cosh(\xi_1 x)+\left[\xi_2^2+\left(\alpha^2-\lambda_2^2\right)\right]\cos(\xi_2 x)\right\}M_1^*(0) \\ &-\frac{\alpha^2\left(\kappa^2-1\right)}{\xi_1^2+\xi_2^2}\left[\cosh(\xi_1 x)-\cos(\xi_2 x)\right]M_2^*(0)+\frac{\alpha^2\left(\kappa^2-1\right)}{\xi_1^2+\xi_2^2}\left[\frac{\sinh(\xi_1 x)}{\xi_1}-\frac{\sin(\xi_2 x)}{\xi_2}\right]V^*(0), \end{aligned} \tag{75}$$

$$
\begin{aligned}
M_2^*(x) = u''(x) = &\left(\frac{1}{\xi_1^2+\xi_2^2}\right)\left\{\left[\xi_1^2-\alpha^2\left(\kappa^2-1\right)\right]\xi_1\sinh\left(\xi_1 x\right)-\left[\xi_2^2+\alpha^2\left(\kappa^2-1\right)\right]\xi_2\sin\left(\xi_2 x\right)\right\}u'(0) \\
&-\left(\frac{\alpha^2}{\xi_1^2+\xi_2^2}\right)\left[\xi_1\sinh\left(\xi_1 x\right)+\xi_2\sin\left(\xi_2 x\right)\right]\theta(0)-\left(\frac{\alpha^2}{\xi_1^2+\xi_2^2}\right)\left[\cosh\left(\xi_1 x\right)-\cos\left(\xi_2 x\right)\right]M_1^*(0) \\
&+\left(\frac{1}{\xi_1^2+\xi_2^2}\right)\left\{\left[\xi_1^2-\alpha^2\left(\kappa^2-1\right)\right]\cosh\left(\xi_1 x\right)+\left[\xi_2^2+\alpha^2\left(\kappa^2-1\right)\right]\cos\left(\xi_2 x\right)\right\}M_2^*(0) \\
&-\left(\frac{1}{\xi_1^2+\xi_2^2}\right)\left\{\left[\xi_1^2-\alpha^2\left(\kappa^2-1\right)\right]\frac{\sinh\left(\xi_1 x\right)}{\xi_1}+\left[\xi_2^2+\alpha^2\left(\kappa^2-1\right)\right]\frac{\sin\left(\xi_2 x\right)}{\xi_2}\right\}V^*(0),
\end{aligned}
\tag{76}
$$

$$V^*(x) = V^*(0), \tag{77}$$

i.e.,

$$M_1^*(x) = t_{42}(x)u'(0)+t_{43}(x)\theta(0)+t_{44}(x)M_1^*(0)+t_{45}(x)M_2^*(0)+t_{46}(x)V^*(0), \tag{78}$$

$$M_2^*(x) = t_{52}(x)u'(0)+t_{53}(x)\theta(0)+t_{54}(x)M_1^*(0)+t_{55}(x)M_2^*(0)+t_{56}(x)V^*(0), \tag{79}$$

$$V^*(x) = V^*(0). \tag{80}$$

Writing in matrix form:

$$
\begin{Bmatrix} u(x) \\ u'(x) \\ \theta(x) \\ M_1^*(x) \\ M_2^*(x) \\ V^*(x) \end{Bmatrix}
=
\begin{bmatrix}
1 & t_{12}(x) & t_{13}(x) & t_{14}(x) & t_{15}(x) & t_{16}(x) \\
0 & t_{22}(x) & t_{23}(x) & t_{24}(x) & t_{25}(x) & t_{26}(x) \\
0 & t_{32}(x) & t_{33}(x) & t_{34}(x) & t_{35}(x) & t_{36}(x) \\
0 & t_{42}(x) & t_{43}(x) & t_{44}(x) & t_{45}(x) & t_{46}(x) \\
0 & t_{52}(x) & t_{53}(x) & t_{54}(x) & t_{55}(x) & t_{56}(x) \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{Bmatrix} u(0) \\ u'(0) \\ \theta(0) \\ M_1^*(0) \\ M_2^*(0) \\ V^*(0) \end{Bmatrix}.
\tag{81}
$$

The transfer matrix is:

$$
\boldsymbol{T}(\boldsymbol{x}) =
\begin{bmatrix}
1 & t_{12}(x) & t_{13}(x) & t_{14}(x) & t_{15}(x) & t_{16}(x) \\
0 & t_{22}(x) & t_{23}(x) & t_{24}(x) & t_{25}(x) & t_{26}(x) \\
0 & t_{32}(x) & t_{33}(x) & t_{34}(x) & t_{35}(x) & t_{36}(x) \\
0 & t_{42}(x) & t_{43}(x) & t_{44}(x) & t_{45}(x) & t_{46}(x) \\
0 & t_{52}(x) & t_{53}(x) & t_{54}(x) & t_{55}(x) & t_{56}(x) \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\tag{82}
$$

For the continuous beam divided into n elements, the relationship between the displacements and the internal forces at the final and initial node of the continuous model is obtained by successively applying the multiplication of transfer matrices [22], i.e.,

$$
\begin{Bmatrix} u(h_n) \\ u'(h_n) \\ M^*(h_n) \\ V^*(h_n) \end{Bmatrix}
= \boldsymbol{T}_n(\boldsymbol{h}_n)\boldsymbol{T}_{n-1}(\boldsymbol{h}_{n-1})\ldots\boldsymbol{T}_3(\boldsymbol{h}_3)\boldsymbol{T}_2(\boldsymbol{h}_2)\boldsymbol{T}_1(\boldsymbol{h}_1)
\begin{Bmatrix} u(0) \\ u'(0) \\ M^*(0) \\ V^*(0) \end{Bmatrix}
= \boldsymbol{T}\begin{Bmatrix} u(0) \\ u'(0) \\ M^*(0) \\ V^*(0) \end{Bmatrix},
\tag{83}
$$

where [22]:

$$\boldsymbol{T} = \boldsymbol{T}_n(\boldsymbol{h}_n)\boldsymbol{T}_{n-1}(\boldsymbol{h}_{n-1})\ldots\boldsymbol{T}_3(\boldsymbol{h}_3)\boldsymbol{T}_2(\boldsymbol{h}_2)\boldsymbol{T}_1(\boldsymbol{h}_1) = \prod_{k=n}^{1}\boldsymbol{T}_k(\boldsymbol{h}_k). \tag{84}$$

5 Numerical examples

To the best of the author's knowledge, there is only one semi-analytical method proposed by Hegedűs and Kollár [43] in the literature that has been utilized by the reference method [48] to solve the stability analysis for global buckling of structural systems and tall buildings using the sandwich beam. Therefore, Section 5 first presents the results of applying the modified transfer matrix method to calculate the critical global buckling load of a coupled shear wall studied by Zalka [48], and then compares the computational costs associated with the modified transfer matrix method (MTMM) and the classical transfer matrix method (TMM). The replacement beam used is the sandwich beam with a transfer matrix size of 6 × 6 and three equivalent stiffnesses (global bending, global shear, and local bending). To establish the boundary conditions of the sandwich beam, the coupled shear wall is modeled as a cantilever beam with perfect fixing at the base, inducing zero displacements and rotations at the base, and zero internal forces at the top of the beam.

5.1 Three-bay coupled shear wall

The studied reference coupled shear wall (Fig. 3) has 30 levels; however, to verify the coincidence between the results, the number of levels will be varied from 5 to 30, resulting in a total of 25 coupled shear walls. Varying the number of levels will indirectly allow us to verify the precision of the proposed method for a wide range of types of behavior (global bending, global shear, local bending, and interaction between them).

The wall thickness is t = 0.30 m, the cross-section of the beams is 0.3 m / 1.0 m in the first and third bay and 0.3 m / 0.5 m in the second bay, the cross-sectional area and the second moment of area of the beams are A_{b1} = 0.3 m^2 and I_{b1} = 0.025 m^4, and A_{b1} = 0.15 m^2 and I_{b1} = 0.03125 m^4, respectively. The modulus of elasticity of the structure is $E = 30 \times 10^6$ kN/m^2, the modulus of elasticity in shear is $G = 12.5 \times 10^6$ kN/m^2 [48] and a uniformly distributed vertical load of 60 kN/m is applied to the connecting beams.

The calculation of the characteristic stiffnesses of the continuous sandwich model is detailed in Zalka's book [48], page 62 Section 2.5.2.

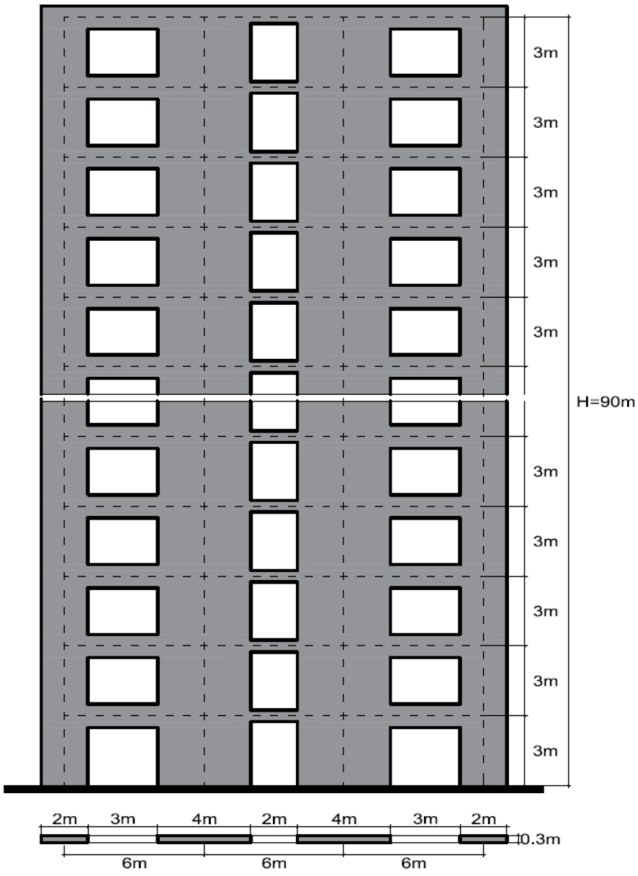

Fig. 3 Three-bay coupled shear wall [48]

The global bending stiffness is:

$$K_{b1} = E\sum_{j=1}^{n} A_{c,j}t_j^2 = 3564000000 \text{ kN m}^2. \tag{85}$$

The global shear stiffness of the coupled shear walls that is associated with the beams is:

$$K_b^* = \frac{6EI_{b,1}\left(\left(l_1^* + s_1\right)^2 + \left(l_1^* + s_2\right)^2\right)}{l_1^{*3}h\left(1 + 12\dfrac{\rho EI_{b,1}}{l_1^{*2}GA_{b,1}}\right)} \times 2 + \frac{6EI_{b,2}\left(\left(l_2^* + s_2\right)^2 + \left(l_2^* + s_3\right)^2\right)}{l_2^{*3}h\left(1 + 12\dfrac{\rho EI_{b,2}}{l_1^{*2}GA_{b,2}}\right)} = 7659041 \text{ kN}. \tag{86}$$

The part of shear stiffness that depends on the wall sections is:

$$K_c = \sum_{j=1}^{n} \frac{\pi^2 EI_{c,j}}{h^2} = 118435253 \text{ kN}. \tag{87}$$

The global shear stiffness is:

$$K_{s1} = K_b^* r^* = 7193826 \text{ kN}, \tag{88}$$

where the reduction factor r^* is:

$$r^* = 0.9393. \tag{89}$$

The local bending stiffness is:

$$K_{b2} = Er^* \sum_{j=1}^{n} I_{c,j} = 101440017 \text{ kN m}^2. \tag{90}$$

Table 1 shows the characteristic stiffnesses of the coupled shear wall calculated by Zalka [48].

Using the equations for the calculation of K_{b1}, K_{s1} and K_{b2} proposed by Zalka [48], the two parameters that describe the behavior of the tall building are expressed as:

$$\alpha = \sqrt{\frac{K_{s1}}{K_{b1}}} = \sqrt{\frac{7193826}{101440017}} = 0.2663, \tag{91}$$

$$k = \sqrt{1 + \frac{K_{b2}}{K_{b1}}} = \sqrt{1 + \frac{101440017}{3564000000}} = 1.0141. \tag{92}$$

To initiate the iterative process using the transfer matrix method, it proves beneficial to first estimate the critical global buckling load using Föppl's theorem. As per Föppl's theorem, the approximate critical global buckling load for such a structure is given by:

$$q_{cr,\text{Föppl's}} \approx \left[\sum_{i=1}^{n} \frac{1}{q_{cr,i}} \right]^{-1}, \tag{93}$$

Table 1 Equivalent stiffnesses of the sandwich beam [48]

Global bending: K_{b1} (kN m2)	Global shear: K_{s1} (kN)	Local bending: K_{b2} (kN m^2)
3564000000	7193826	101440017

where:

$$q_{cr,i} = \frac{\pi^2 K_{b2,i}}{4H^2} + \left\{ \left[\frac{\pi^2 K_{b1,i}}{4H^2} \right]^{-1} + K_{s1,i}^{-1} \right\}^{-1}. \tag{94}$$

Fig. 4 show the calculation of the critical global buckling load using Zalka's method [48], the proposed method, and Föppl's theorem. Excellent agreements are observed between Zalka's method and the proposed method. Furthermore, Föppl's theorem has proven to be an excellent tool, offering a straightforward lower limit to initiate iteration with excellent convergence. The term ratio refers to the relationship between the critical global buckling load and the total vertical load. It is widely used in structural engineering practice as a reference parameter to evaluate the overall performance of the structure.

Using the Zalka method [48] as a reference, Fig. 5 show the percentage of error in the calculation of the critical global buckling load using the proposed method and Föppl's theorem. As expected, the critical load estimation using Föppl's theorem is only applicable for a first estimate and starting the iterations of the associated numerical method. However, the proposed method leads to a minimum error of −4.07%. An interesting observation is the chain-like behavior exhibited by the error profiles of both methods when compared to the reference method [48]. This is because the reference method only provides approximate results for specific cases, necessitating interpolation for intermediate cases.

It is important to demonstrate the efficiency of the proposed method (MTMM) compared to the classical method (TMM). Therefore, Fig. 6 uses the data from the numerical example in Section 5.1 concerning the three-bay coupled shear wall to illustrate the comparison of normalized execution time relative to the maximum time of the proposed method for an iteration of the eigenvalue problem associated with the global buckling problem. For calculating

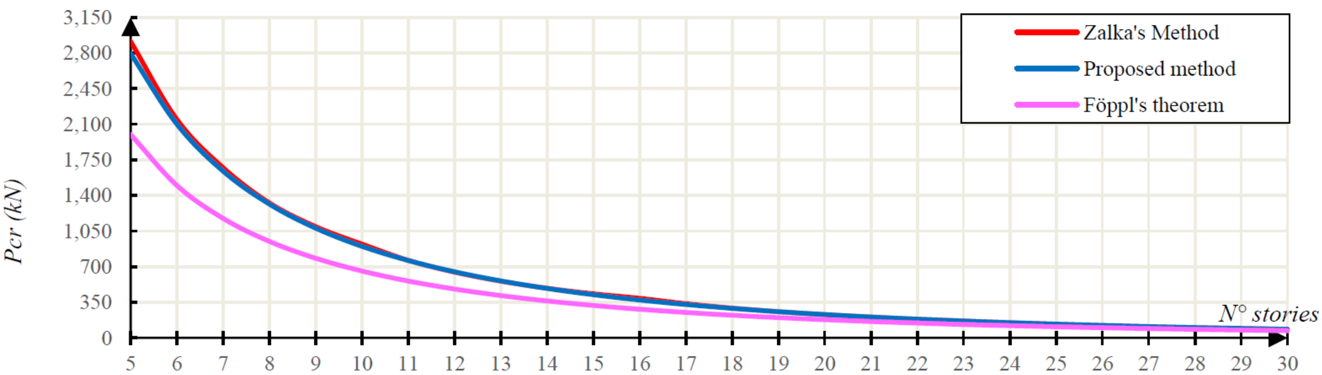

Fig. 4 Critical global buckling loads obtained using Zalka's method [48], Föppl's theorem, and the proposed method

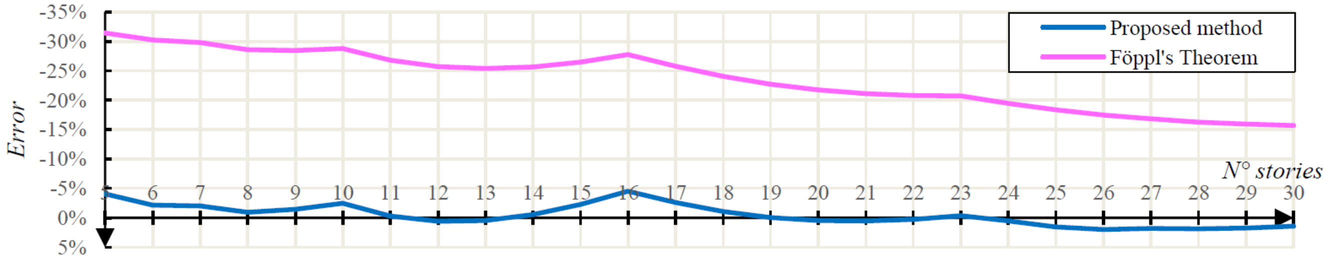

Fig. 5 Accuracy in the calculation of the critical global buckling load using the proposed method and Föppl's theorem

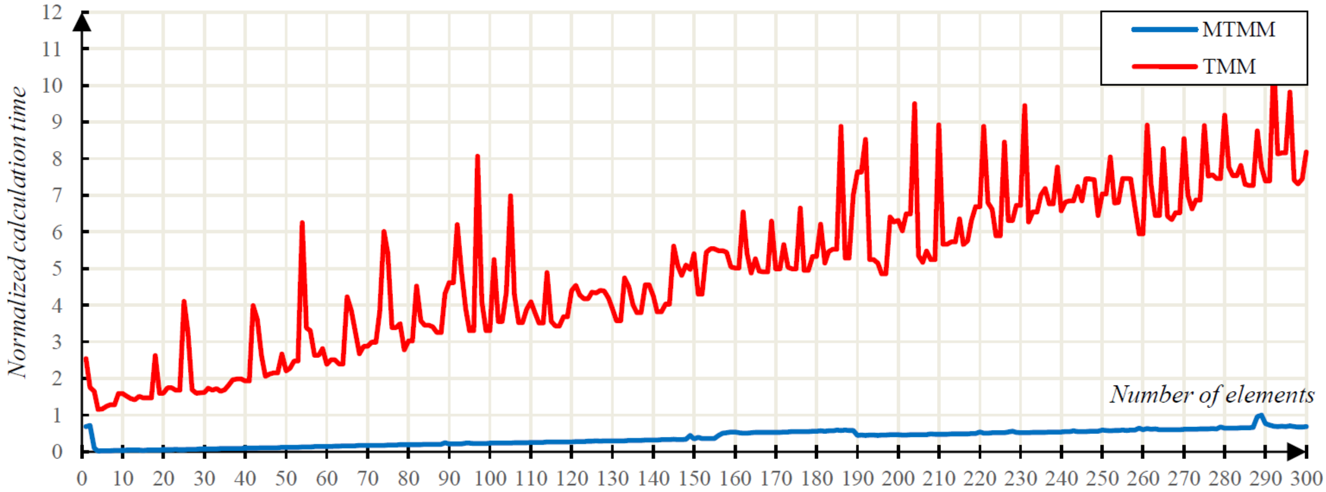

Fig. 6 Comparison of computation costs in normalized execution time using MATLAB [49]

the execution time of both the proposed and traditional methods, a code generated in the MATLAB numerical computing system was used [49]. A significant reduction in computation time is observed, which increases linearly as the number of discretized elements grows. In other words, as we increase the number of discretized elements, the proposed method reduces computation time, thereby decreasing computational cost and simultaneously reducing computer memory usage.

6 Conclusions

This paper aims to address one of the two main challenges of the classical transfer matrix method in solving the global elastic stability problem of uniform and variable beams, specifically the calculation of the inverse of the zero matrix. The primary contribution is the analytical derivation of the terms of the transfer matrix, thereby eliminating the need to compute the inverse of the zero-matrix used in the classical method and significantly reducing computational cost.

For this purpose, the Laplace transform was applied to the equilibrium equations. Then, the inverse Laplace transform was applied to the displacements, resulting in direct expressions for displacements (at an arbitrary position) in terms of the displacements and internal forces evaluated at the zero position (with internal forces obtained directly from the displacements). Consequently, the matrix that transforms displacements and internal forces at the zero position to displacements and internal forces at an arbitrary position is the transfer matrix.

An application to a three-bay coupled shear wall studied in the literature [48] was presented to compare the results of the transfer matrix method with the semi-analytical method based on the continuous method [48]. The results show good agreement between both methods (even for a few stories), validating the application of the transfer matrix method in the analysis of structures using the classical sandwich beam. Finally, the computation time between the modified transfer matrix method (MTMM) and the classical method (TMM) was compared. The results indicate that the proposed method reduces computation time with a linear increase as the number of discretizations increases. This drastic reduction in computation time will be even more beneficial in the successive iterations required to solve the eigenvalue problem associated with the global buckling problem.

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Nomenclature

$u(x)$	Lateral displacement
$\theta(x)$	Rotational displacement
x	Longitudinal axis of the beam
P	Vertical load distributed along the length of the beam
K_b	Bending stiffness of the Timoshenko beam
K_s	Shear stiffness of the Timoshenko beam
K_{b1}	Global bending stiffness of the sandwich beam
K_{s1}	Global shear rigidity of the sandwich beam
K_{b2}	Local bending rigidity of the sandwich beam
$M(x)$	Bending moment
$V(x)$	Shear force
λ_1	Parameter that relates the lateral load to the shear stiffness of the Timoshenko beam
λ_2	Parameter that relates the lateral load to the local bending stiffness of the sandwich beam
$U(s)$	Laplace transform of lateral displacement
$\Phi(s)$	Laplace transform of rotational displacement
$T(x)$	Transfer matrix of the i-th element
β	Parameter that relates the shear and bending stiffness of the Timoshenko beam
α	Parameter that relates the global shear and local bending stiffness of the sandwich beam
κ	Parameter that relates the local bending and global bending stiffness of the sandwich beam
E	Elastic modulus
G	Shear modulus
ν	Poisson's coefficient
L	Total beam length

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