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Frequencies of Near Regular Structures Using the Results of the Corresponding Regular Structures

Ali Kaveh[1*], Nabi Khazaee[1]

[1] School of Civil Engineering, Iran University of Science and Technology, PO Box 16846-13114, Iran
[*] Corresponding author, e-mail: alikaveh@iust.ac.ir

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Abstract

In this paper a method is presented for calculating the eigenvalues of perturbed matrices corresponding to their initial and unperturbed state. In other word, instead of solving the eigenvalue problem of perturbed matrix with size (n x n), only it is sufficient to solve eigenvalue problem for a matrix with dimension (m x m) where m is less than n and their difference (n – m) is considerable. By means of this method, eigenvalues and frequencies of near regular structures considering those of the corresponding regular structures are calculated.

Keywords

perturbed matrices, near regular structures, reanalysis problem, eigenvalue problem, structural frequencies, modal analysis

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1 Introduction

Suppose we have an initial matrix $[A]$ which associated eigenvalues and eigenvectors are assumed to be known as (λ_1, λ_2, λ_3,..,λ_n) and (v_1, v_2, v_3,..,v_n). However, suppose for some reasons this matrix perturbs.

One way to find the eigenvalues of the perturbed matrix is to find this property by the common procedure and using the following Eq. (1): (A_p = perturbed matrix).

$$\det\left(A_p - \lambda I\right) = 0 \tag{1}$$

Even though the appearance of Eq. (1) is simple, in many situations using this formula costs a lot of computational time and also requires high storage. Therefore, it would be logical, by some changes in eigenvalues of the unperturbed matrix, required eigenvalues for perturbed one could be calculated; For this aim several procedures by researchers have been performed before. In these methods special conditions are required for both the perturbed matrix and delta matrix (delta matrix is the difference between the perturbed and unperturbed matrix). Videlicet, if you find a way to write the delta matrix as $\sigma\overline{u}\overline{v}^T$, vectors $\overline{u}$ and $\overline{v}$ must have special entries. For example, Horn and Serra-Capizzano [1], supposed the perturbed matrix to be in the form $cA + (1-c)x\lambda v^T$ where A is the unperturbed matrix and nonzero complex vectors x and v satisfy $Ax = \lambda x$ and $v^Tx = 1$ also c is any complex number, or in another example Brauer [2] has supposed the perturbed matrix to be in the form $(A + xv^T)$ where A is the unperturbed matrix that for any complex vectors x satisfies $Ax = \lambda x$. As it can be seen, all of these methods can apply only on special perturbed matrices.

An another attempt for calculating eigenvalues of the perturbed matrices have carried out by Bamieh [3]; where has been presumed perturbed matrix could be written in the form $A_0 + \varepsilon A_1$.

Careful study of these methods, one finds out that either the perturbed matrices and their associated delta matrices should have special patterns.

In present paper we have generalized the theorem described by Ding and Yao [4]; This theorem like other mentioned methods is just for special situations. However, by this generalization, eigenvalues of any perturbed matrices based on their unperturbed matrices can obtain with very low error. eventually, by this generalization, eigenvalues of the perturbed matrices by size ($n \times n$) can be reduced to finding eigenvalues of one matrix of size ($m \times m$), where m is lower than n and their difference ($n - m$) is considerable. This generalization is only applicable for matrices which unperturbed form is symmetric.

In the following, the theorem described by Ding and Yao [4] has been presented:

Let $[A]$ be an $n \times n$ real matrix with eigenvalues (λ_1, λ_2, λ_3, ..., λ_n) counting algebraic multiplicities, and for $1 \le k \le n$. Let $U = (u_1, u_2, u_2, ..., u_k)$ and $V = (v_1, v_2, v_2, ..., v_k)$ be

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real column vectors such that $(v_1, v_2, v_2, ..., v_k)$ are linearly independent left eigenvectors of $[A]$ corresponding to the eigenvalues $(\lambda_1, \lambda_2, \lambda_2, ..., \lambda_k)$, respectively. Then the eigenvalues of the matrix $B = A + \sum_{i=1}^{k} u_i v_i^T$ are $(\mu_1, \mu_2, \mu_2, ..., \mu_k, \lambda_{1+k}, \lambda_{2+k}, \lambda_{2+k}, ..., \lambda_n)$ where $(\mu_1, \mu_2, \mu_2, ..., \mu_k)$ are the eigenvalues of the $k \times k$ matrix $diag(\lambda_1, \lambda_2, \lambda_2, ..., \lambda_k) + U^T V$.

As it can be seen, for using the theorem of Ding and Yao [4], delta matrix must have special format. To rephrase it, vector v_i^T in delta matrix $(\sum_{i=1}^{k} u_i v_i^T)$ must be left eigenvectors of the unperturbed matrix and has not arbitrary format. About error associated to this theorem, it is worth mentioning, if all conditions specially that of corresponding to vector v_i^T satisfy the requirements of the theorem, we will not accost with any error. But the problem arise here that a lot of systems exist which associated perturbations are arbitrary. Hence, by indiscriminately using of this theory for arbitrary perturbed systems, we will face with large errors. In our paper, we generalized this delta matrix such that v_i^T can be an arbitrary vector; Consequently, calculation of the periods of near regular structures can then be evaluated.

In structural mechanics, analysis of perturbed systems has been considered as one of essential tasks. Usually, two types of perturbation to initial systems have been applied for analysis, one of them is topology changes such adding or omitting elements to initial structure [5, 6]; Other types of structures are associated to Parameter modification involves modifying structural parameters like cross-sectional area, mass and material elastic rigidity, etc., with boundary shapes and topology unchanged [7–10]. In the field of mechanical reanalysis technics, sometimes, the perturbed structure is denoted by near regular structure. [11, 12].

In this paper we will analysis the Perturbed structures (Near Regular) which associated changes do not alter degree of freedom of initial structure.

Since matrices associated to near regular structures often have not well-known patterns, the matrix analysis like inversing or generalized eigenproblems become hard tasks.

In the literature, different numerical based methods, namely the homotopy perturbation techniques [13], Taylor series expansion [14], Padé approximants [15, 16], projection and reduced basis [17], have already been used to approximate perturbed solutions and can be viewed as reanalysis techniques. Also, application of elegant matrix row and column operations on matrices correspond to near regular structure have been performed by Kaveh et al. [18]. Analysis of near regular structures also can be performed in an efficient way using singular value decomposition (SVD) of existing equilibrium in structures [19–21].

2 Method to solve eigenproblem of *q* update matrix

Consider one perturbed matrix which has been deviated from initial and unperturbed form as following: Eq. (2). q update has been denoted in indices of below Eq. (2)).

$$A_p = A + \sigma_1 \overline{u_1}\,\overline{v_1}^T + \sigma_2 \overline{u_2}\,\overline{v_2}^T + \ldots + \sigma_q \overline{u_q}\,\overline{v_q}^T , \quad (2)$$

where, σ is a number and $\overline{u}$ and $\overline{v}$ are column vectors.

Since the purpose of this section is to generalize the theorem of Ding and Yao [4] and find the eigenvalues of the perturbed matrix using already known eigenvalues and eigenvectors of unperturbed matrix, first of all, we should recognize best well-known matrix corresponding to our initial matrix; For applying the effect of changes, this unperturbed matrix must have the following properties:

It must be symmetric. (This assumption is for guaranteeing the independence of the eigenvectors, actually, for method which will describe later, these independence vectors are needed.)

For efficient analysis, this matrix must be well-known matrix. Therefore, its eigenvalues and eigenvectors must be calculated with low computational complexity. (Note: in our method, term "well-known matrix" can be referred to a matrix which eigenvalues and eigenvectors are already known.)

By these assumptions, the delta matrix is defined as below (Eq. (3)):

$$delta = A_p - A . \quad (3)$$

In this step, we need to write our delta matrix as a linear combination of some rank one matrix: Eq. (4).

$$delta = \sigma_1 \overline{u_1}\,\overline{v_1}^T + \sigma_2 \overline{u_2}\,\overline{v_2}^T + \ldots + \sigma_q \overline{u_q}\,\overline{v_q}^T \quad (4)$$

This task can easily be performed by using truncated singular value decomposition (*SVD*) or using the following simple procedure which has been described by a small example:

A *matrix*(*n by n*) where its entry of the *i*th row and *j*th column is σ, is given. Therefore, this matrix can be written as: Eq. (5) (Note: this multiplication is as same as ordinary matrix multiplication.)

$$\sigma e_i e_j^T , \quad (5)$$

where e_i is a column vector which *i*th entry is equal to 1 and other entries are 0. Similarly, e_j is a column vector which *j*th entry is equal to 1 and other entries are 0.

Now, we back to our problem; By considering already known eigenvectors of matrix $[A]$ as $v = (v_1, v_2, v_2, ..., v_n)$. We assume that each of $\overline{v_k}\left(for\, k = 1,2,3\ldots q\right)$ appearing in Eq. (4) could be written as a linear combination of eigenvectors of matrix $[A]$.

Based on this assumption, only symmetric matrices must be chosen for unperturbed matrix. Actually, if a vector (with size n) may be written as linear combination of n other vectors (with size n), those vectors must be independent. Since eigenvectors of symmetric matrices are both independent and orthogonal, they make an assortment of basis for other vectors with same size to be written as a linear combination of those basis. (Note: the fundamental concept of Section 4 is to convert symmetric generalized eigen problem to ordinary one in a way that matrices remain symmetric; Also, further discussion about matrices associated to modal analysis of structures will be presented.)

Therefore, for example for $\overline{v_1}$ we have: Eq. (6).

$$\overline{v_1} = \dot{\beta}_{1,1}v_1 + \dot{\beta}_{1,2}v_2 + \dot{\beta}_{1,3}v_3 + \ldots + \dot{\beta}_{1,n}v_n \tag{6}$$

Now, suppose some of $\dot{\beta}_{1,k}$'s (for $k = 1,2,3,...,n$) are zero, therefore by considering remaining terms we can write $\overline{v_1}$ as m_1 eigenvectors of symmetric matrix $[A]$ as below: Eq. (7). (Note: The fundamental concept of Section 3 is to represent a procedure assuring us that some of $\dot{\beta}_{1,k}$'s become zero. However, if none are zero, the remaining terms are the same as before).

$$\overline{v_1} = \beta_{1,1}v_{i_1^1} + \beta_{1,2}v_{i_2^1} + \beta_{1,3}v_{i_3^1} + \ldots + \beta_{1,m_1}v_{i_{m_1}^1}\ , \tag{7}$$

where, subscripts of Eq. (7) are as following ($m_j = 1,2,...,m_1$): Eq. (8).

$$i_{m_j}^1 = one\ number\ from\ set\left[1,2,\ldots n\right] - \left[i_1^1\right] - \left[i_2^1\right] - \ldots - \left[i_{m_j-1}^1\right] \tag{8}$$

Similarly, other $\overline{v_k}\left(for\,k = 2,3\ldots q\right)$ appearing in Eq. (4) could be written as a linear combination of m_k($fork = 2,3,...,q$) eigenvectors of matrix $[A]$ where m_k is a number from set (1,2,3,...,n). Eq. (9) and Eq. (10).

$$\overline{v_k} = \dot{\beta}_{k,1}v_1 + \dot{\beta}_{k,2}v_2 + \dot{\beta}_{k,3}v_3 + \ldots + \dot{\beta}_{k,n}v_n \tag{9}$$

$$\overline{v_k} = \beta_{k,1}v_{i_1^k} + \beta_{k,2}v_{i_2^k} + \beta_{k,3}v_{i_3^k} + \ldots + \beta_{k,m_k}v_{i_{m_k}^k} \tag{10}$$

Since ($v_{i_1^1}, v_{i_2^1}, v_{i_3^1}, \ldots, v_{i_{m_1}^1}$) are the eigenvectors of symmetric matrix $[A]$, therefore: Eq. (11) and Eq. (12).

$$for\ k = t \rightarrow {v_{i_k^1}}^T v_{i_t^1} = {v_{i_t^1}}^T v_{i_k^1} = 1 \tag{11}$$

$$for\ \ k \neq t \rightarrow {v_{i_k^1}}^T v_{i_t^1} = {v_{i_t^1}}^T v_{i_k^1} = 0 \tag{12}$$

Hence, $\beta_{1,1}, \beta_{1,2},..., \beta_{1,m_1}$ can be obtained by the following formulae: Eq. (13)

$$\beta_{1,1} = {v_{i_1^1}}^T\overline{v_1}, \beta_{1,2} = {v_{i_2^1}}^T\overline{v_1}, \ldots, \beta_{1,m_1} = {v_{i_{m_1}^1}}^T\overline{v_1} \tag{13}$$

Similarly, for other vectors we have: Eq. (14).

$$\beta_{k,1} = v_{i_1^k}^T v_k, \beta_{k,2} = v_{i_2^k}^T v_k, \ldots, \beta_{k,m_k} = v_{i_{m_k}^k}^T\ v_k \tag{14}$$

By these coefficients, term "$\sigma_1\overline{u_1 v_1}^T$" is: Eq. (15) and Eq. (16).

$$\sigma_1\overline{u_1 v_1}^T = \sigma_1\overline{u_1}\left(\beta_1 {v_{i_1^1}}^T + \beta_2 {v_{i_2^1}}^T + \ldots + \beta_{m_1}{v_{i_{m_1}^1}}^T\right) \tag{15}$$

$$\sigma_1\overline{u_1 v_1}^T = \gamma_{1_1}\overline{u_1}{v_{i_1^1}}^T + \gamma_{1_2}\overline{u_1}{v_{i_2^1}}^T + \ldots + \gamma_{1_{m_1}}\overline{u_1}{v_{i_{m_1}^1}}^T = \sum_{g=1}^{g=m_1}\gamma_{1_g}\overline{u_1}{v_{i_g^1}}^T \tag{16}$$

Similar task can be applied to $\sigma_2\overline{u_2 v_2}^T$, $\sigma_3\overline{u_3 v_3}^T$,..., $\sigma_q\overline{u_q v_q}^T$; hence, generally we have following: Eq. (17).

$$\begin{aligned}\sigma_q\overline{u_q v_q}^T &= \gamma_{q_1}\overline{\mathrm{u_q}}{v_{i_1^q}}^T + \gamma_{q_2}\overline{u_q}{v_{i_2^q}}^T + \ldots + \gamma_{q_{m_q}}\overline{u_q}{v_{i_{m_q}^q}}^T \\ &= \sum_{g=1}^{g=m_q}\gamma_{q_g}\overline{\mathrm{u_q}}{v_{i_g^q}}^T\end{aligned} \tag{17}$$

Finally, by putting derived terms in Eq. (2) we have: Eq. (18).

$$A_p = A + \sum_{g=1}^{g=m_1}\gamma_{1_g}\overline{u_1}{v_{i_g^1}}^T + \sum_{g=1}^{g=m_2}\gamma_{2_g}\overline{u_2}{v_{i_g^2}}^T + \ldots + \sum_{g=1}^{g=m_q}\gamma_{q_g}\overline{\mathrm{u_q}}{v_{i_g^q}}^{\mathrm{T}} \tag{18}$$

Since some of ${v_{i_g^1}}^T$, ${v_{i_g^2}}^T$,..., ${v_{i_g^q}}^{\mathrm{T}}$ can be identical, by factorizing the same arguments, we obtain: Eq. (19).

$$A_p = A + \breve{u}_1{v_{j_1}}^T + \breve{u}_2{v_{j_2}}^T + \breve{u}_3{v_{j_3}}^T + \ldots + \breve{u}_{m_t}{v_{j_{m_t}}}^T\ , \tag{19}$$

where ${v_{j_{m_t}}}^T$ is the m_t'th identical arguments which factorized coefficient is $\breve{u}_{m_t}$.

Now, everything is prepared to be used in the aforementioned theorem. Videlicet, instead of finding the eigenvalues for a matrix by size $n \times n$, only finding the eigenvalues of the following matrix with the size $m_t \times m_t$ is needed. Eq. (20)

$$diag\left(\lambda_{j_1}, \lambda_{j_2}, \lambda_{j_3}, \ldots, \lambda_{j_{m_t}}\right) + \begin{bmatrix}\breve{u}_1^T \\ \breve{u}_2^T \\ \breve{u}_3^T \\ \vdots \\ \breve{u}_{m_t}^T\end{bmatrix}\begin{bmatrix}v_{j_1} & v_{j_2} & v_{j_3} & \cdots & v_{j_{m_t}}\end{bmatrix} \tag{20}$$

3 How to find m_1, m_2,..., m_q to calculate m_t?!!

Each vector with the size n can be written as a linear combination of n linear independent basis. Therefore, since vectors $\overline{v_1}, \overline{v_2}, \overline{v_3}, \ldots, \overline{v_q}$ are ($n \times 1$), one can write them as the linear combination of all eigenvectors of symmetric matrix [A]. However, by using this combination, and regarding Eq. (20) we will struggle with a matrix where the size is same as that of the initial matrix [A_p].

Instead of this task, we try to write the vectors $\overline{v_1}, \overline{v_2}, \overline{v_3}, \ldots, \overline{v_q}$ as a linear combination of some but not all of eigenvectors of symmetric matrix [A]. Actually, we want to write $\overline{v_k}\ (for\ k = 1,2,3\ldots q)$ appearing in Eq. (4) as a linear combination of m_k(for k = 1,2,3,...,q) eigenvectors of matrix [A]. For this purpose, we will try to find out, for example for $\overline{v_1}$, which of n basis is perpendicular to $\overline{v_1}$? , if we find such basis, our vector $\overline{v_1}$ is not depends on those vectors. Hence, $\overline{v_1}$ can be written as a linear combination of other bases excluding that dependent basis to our vector $\overline{v_1}$. Eventually, for instance $\boldsymbol{m}_1$ basis (instead of whole n basis) for $\overline{v_1}$ can be found as the following: first of all, we find a row matrix denoted by $R_1 = \overline{v_1}^T v$. Here v is a matrix containing the eigenvectors of the symmetric matrix [A] as $v = v_1, v_2, v_3, \ldots v_n$. Consequently, if kth entry of the row matrix R_1 is zero, this mean our vector $\overline{v_1}$ is not depends on the kth eigenvector of the symmetric matrix [A]. Hence, by separating those eigenvectors depending on our vector $\overline{v_1}$, we have $\boldsymbol{m}_1$ basis (instead of whole n basis) to write our vector $\overline{v}$ as a linear combination of those.

Similar task can be accomplished for $\overline{v_2}, \overline{v_3}, \ldots, \overline{v_q}$ to obtain $\boldsymbol{m}_2, \ldots, \boldsymbol{m}_q$ basis from eigenvectors of the symmetric matrix [A].

By mentioned procedure the following question will arise because most of the time the $\overline{v_t}^T v\left(for\, t = 1,2,3,\ldots,q\right)$ has not zero entry. In another word, for $\overline{v}\ v$ becoming a row matrix containing zero entries, each eigenvector of matrix [A] must have sufficient zero entries, and this situation can happen only for some very special matrices.

For overcoming this problem, by accepting some very small errors, instead of finding independent basis to our vector (for example $\overline{v_1}$) by formula $\overline{v_1}^T v_s = 0$ (Note: v_s is a vector from previous defined set v), we can use the following formula: Eq. (21) (Note: Actually, term ε is supplanted with 0 in $\overline{v_1}^T v_s = 0$).

$$0 < \overline{v_1}^T v_s \leq \varepsilon \tag{21}$$

By using Eq. (21), we could say two column vectors such as x and y are independent if their multiplication ($x^T y$) be less than (ε) instead of being equal to zero (0).

Up to now, no algorithm has been found to specify one good quantity for (term ε), however, error is reducing by obtaining smaller term "ε" and matrix dimension is reducing through obtaining larger term "ε". For instance, referring to second example of this paper, by accepting average error equals to 0.11 *percentage*(%) we can use matrix with size 1348 instead of initial matrix size which was 2184. By another calculation, with accepting average error as 1.62 *percentage*(%) we can use matrix with size 1343 instead of initial matrix size which was 2184. Also, for the second and third examples, the variation of *error* (%) and matrix dimension (instead of initial matrix size) through increasing the term ε, are provided in Sections 6 and 7.

4 Application for finding periods of near regular structures

Modal analysis specially for perturbed systems is one of the essential tasks required in structural mechanics [21–29], For structures with massive degree of freedom specially in finite element method, this calculation face with two big matrices denoted by mass matrix and stiffness matrix. For this approach, the solution of the following eigenvalue problem is needed: Eq. (22).

$$Kx = \lambda Mx\ , \tag{22}$$

where K and M are the stiffness matrix and mass matrix, respectively, and λ's are eigenvalues corresponding to their eigenvector denoted by x.

Solving the mentioned generalized eigenproblem always has been one of time and storage consuming operation during matrix structural analysis. In this paper, the problem has been solved for the near regular structures by using modal analysis of the corresponding regular structure.

For applying the effect of the changes, we have: Eq. (23) (R = regular, NR = near regular).

$$K_{NR} x_{NR} = \lambda_{NR} M_{NR} x_{NR} \tag{23}$$

In this step we need to reduce the generalized eigenproblem to an ordinary eigenproblem. This task can be done by multiplying both sides of Eq. (23) by M_{NR}^{-1}, so we have: Eq. (24).

$$M_{NR}^{-1} K_{NR} x_{NR} = \lambda_{NR} x_{NR} \tag{24}$$

By using the same task for corresponding regular structure, we have: Eq. (25).

$$K_R x_R = \lambda_R M_R x_R \rightarrow M_R^{-1} K_R x_R = \lambda_R x_R \tag{25}$$

As was mentioned before, unperturbed matrix must be symmetric. Therefore, in the process of finding M_R^{-1} and multiplying it with K_R, the result $M_R^{-1} K_R$ must be symmetric; Mass matrix of any structures is symmetric, eventually, its inverse M_R^{-1} is also symmetric. Since the stiffness matrix of any structures is symmetric, the only condition for $M_R^{-1}K_R$ to be symmetric is that M_R^{-1} and K_R must commute. (Note: two matrices such U and W are said to commute if $UW = WU$).

The following part will denote conditions for M and K to be qualified using method of this paper: (this part can even be used for generalized eigen problem associated to non-modal analysis problems).

If M and K are both symmetric circulant matrices, since M^{-1} is symmetric circulant and also this fact that circulant matrices commute with each other, generalized eigen problem can be reduced to symmetric ordinary eigen problem below: Eq. (26).

$$M_{NR}{}^{-1}K_{NR}x_{NR} = \lambda_{NR}x_{NR} \text{ and } M_R{}^{-1}K_Rx_R = \lambda_Rx_R \tag{26}$$

If M is diagonal matrix with same values on its diagonal and K is symmetric matrix, since $M^{-1}K$ is symmetric, generalized eigen problem can reduced to symmetric ordinary eigen problem given in Eq. (26)

If M is diagonal or non-diagonal symmetric matrix and K is symmetric matrix, by some little operations Eq. (22) converts to ordinary symmetric one Eq. (27): Chopra [22].

$$\left(M^{-0.5}KM^{-0.5}\right)\left(M^{0.5}x\right) = \lambda\left(M^{0.5}x\right) \tag{27}$$

So, we define below matrices: Eq. (28).

$$D = M^{-0.5}KM^{-0.5} \text{ and } y = M^{0.5}x \tag{28}$$

Eventually, our problem converted to symmetric ordinary eigen problem $Dy = \lambda y$ where D is symmetric. (Note: eigenvectors of $M_R^{-0.5}K_RM_R^{-0.5}$ are the same as $M^{0.5}w$ (w is a matrix containing eigenvectors of $K_Rx_R = \lambda_RM_Rx_R$).

In this step one question will arise upon efficient way for finding (M^{-1}, $M^{-0.5}$, $M^{0.5}$); It is worth mentioning, this problem can only occur for near regular ones. In another word, since our chosen unperturbed structure is regular, calculation of (M_R^{-1}, $M_R^{-0.5}$, $M_R^{0.5}$) can be done very easily. (Note: for diagonal matrices, this calculation does not have any hard task; for those matrices, to find M^{-1}, it is sufficient to inverse their diagonal elements and for finding $M^{0.5}$ and $M^{-0.5}$ use square root and root of (–0.5) of their diagonal elements, respectively.)

For overcoming the calculation of terms (M_{NR}^{-1}, $M_{NR}^{-0.5}$, $M_{NR}^{0.5}$), one should consider following question that how matrix M_{NR} deviate from M_R?; Answer of this question is divided to two categories:

First: Near regular structures which can be transformed to regular one by adding or omitting some elements (without changing degree of freedom), and second: Near regular structures which can be transformed to regular one by changing some properties like density.

Since mass matrices for the first category do not change considerably, we can use M_R^{-1} instead of M_{NR}^{-1}.

For the second category, pattern of mass matrix will not change and only entries of cells of mass matrix will change. Since our chosen unperturbed structure is regular and its associated matrices are well-known or have less intricated patterns, we can again use properties of these types of matrices for calculating the terms (M_{NR}^{-1}, $M_{NR}^{-0.5}$, $M_{NR}^{0.5}$). (Note: For diagonal mass matrix if one entry contains zero, for deviating from singularity, it is sufficient to consider one very small number instead of that zero; For other types of mass matrices, it is sufficient to consider mass matrix as $M_{new} = M_{old} + \tau I$ where τ is a very small number.)

Now everything has been prepared for using the method of this paper; In this step, by following the described method, one finds the eigenvalues of $K_{NR}x_{NR} = \lambda_{NR}M_{NR}x_{NR}$ by using eigenvalues and eigenvectors of the regular structure through one of following equations: Eqs. (29) and (30).

$$M_{NR}{}^{-1}K_{NR} = M_R{}^{-1}K_R + delta \tag{29}$$

$$M_{NR}{}^{-0.5}K_{NR}M_{NR}{}^{-0.5} = M_R{}^{-0.5}K_RM_R{}^{-0.5} + delta \tag{30}$$

Since matrices $M_{NR}^{-1}K_{NR}$ and $M_R^{-1}K_R$ are very close to each other and also because of closeness of $M_{NR}^{-0.5}K_{NR}M_{NR}^{-0.5}$ and $M_R^{-0.5}K_RM_R^{-0.5}$, the *delta* matrix is sparse and can be written as some rank one matrices, in which due to the sparsity of the *delta* matrix, this number is considerably smaller than the size of the initial matrix.

5 Example 1

We have an unperturbed matrix as Eq. (31)

$$A = \begin{bmatrix} D_1 & E & z & z & z \\ E & D_2 & E & z & z \\ z & E & D_3 & E & z \\ z & z & E & D_4 & E \\ z & z & z & E & D_5 \end{bmatrix} \tag{31}$$

For some reasons, this matrix converts to perturbed one by $D_1 \to F$ and $D_5 \to G$, Where the matrices D and E and also F and G are as: (Note = z is zero matrix) Eq. (32) to Eq. (35).

$$D_1 = \begin{bmatrix} 1788.451 & 1695.7 \\ 1695.7 & 2074.2 \end{bmatrix}, D_2 = \begin{bmatrix} 1082.579 & 200 \\ 200 & 963.3687 \end{bmatrix} \tag{32}$$

$$D_3 = \begin{bmatrix} 1682.579 & 200 \\ 200 & 1063.369 \end{bmatrix}, D_4 = \begin{bmatrix} 3282.579 & 100 \\ 100 & 1263.369 \end{bmatrix} \quad (33)$$

$$D_5 = \begin{bmatrix} 3688.451 & 622.3003 \\ 622.3003 & 1884.2 \end{bmatrix}, \mathrm{E} = \begin{bmatrix} -280 & 0 \\ 0 & 0 \end{bmatrix} \quad (34)$$

$$F = \begin{bmatrix} 941.2897 & 892.4735 \\ 892.4735 & 1091.684 \end{bmatrix}, G = \begin{bmatrix} 1941.29 & 327.5265 \\ 327.5265 & 991.6843 \end{bmatrix} \quad (35)$$

Now by considering delta matrix of this example, this matrix can be written as a linear combination of 4 rank one matrix; This decomposition can easily be adopted from the truncated singular value decomposition (SVD). Eq. (36)

$$delta = \sigma_1 \overline{u_1 v_1}^T + \sigma_2 \overline{u_2 v_2}^T + \sigma_3 \overline{u_3 v_3}^T + \sigma_4 \overline{u_4 v_4}^T \quad (36)$$

Now, we try to write $\overline{v_1}$ as a linear combination of some but not all of eigenvectors of [A]:

We recall the criteria obtained for finding suitable eigenvectors for basis; For this example, we used $\varepsilon = 0.2$.

By choosing denoted criteria, $\sigma_1 \overline{v_1}$, $\sigma_2 \overline{v_2}$, $\sigma_3 \overline{v_3}$ and $\sigma_4 \overline{v_4}$ are: Eq. (37) to Eq. (39)

$$\sigma_1 \overline{v_1} = 1704.368 v_1 + 689.187 v_3 \quad (37)$$

$$\sigma_2 \overline{v_2} = -1715.759 v_2 \text{ and } \sigma_3 \overline{v_3} = 793.998 v_5 \quad (38)$$

$$\sigma_4 \overline{v_4} = 23.655 v_9 + 104.891 v_{10} \quad (39)$$

Based on decomposition of delta, 4th, 6th, 7th, and 8th eigenvalues of perturbed matrix (A_p) is the same with 4th, 6th, 7th, and 8th eigenvalues of unperturbed matrix [A]. Other eigen values will obtain from one (6 × 6) matrix using Eq. (20).

Comparison between the procedure of this paper and the ordinary method to calculate eigenvalues is as follows Table 1.

6 Example 2

A near regular structure is shown in Fig. 1, for this structure all elements are *IPE*200 except elements of last floor which are *IPE*160, each span is 4 m length and 3 m length. Elasticity modules is assumed to be 19.6123 × 10^{10} N/m^2. And density is assumed to be 76981.81 N/m^3 (7850 Kg/m^3). Size of mass and stiffness matrix of this structure is 2184 × 2184.

By considering correspond regular structure as same configuration of near regular structure, which its all elements are *IPE*200, calculation of frequencies (Note: we calculated square of these numbers) are needed.

Since one of application of present paper is for analyzing perturbed systems, it can be applied for structures which modal analysis is already known; now, we suppose that this system by some reasons is changed to near regular structure, eventually we used already known modal information of unperturbed system to approximately find modal information of perturbed system. Finally, the term "regular structure" in this example refer to a structure that its modal information is available erstwhile, and we want to use this information to find frequencies of near regular structure.

For this structure, we used 2 described criteria ($\varepsilon = 10^{-6}$ and $\varepsilon = 10^{-5}$), and for each of criteria, size of matrix for calculation instead of initial matrix size (initial was 2184) and first 3 modes are denoted in Table 2. Also, exact solution of first 3 modes is delighted and relative error in percentage (%) is reported. Based on Table 2, by accepting average error equals to 0.11 percentage (%) we can find our

Table 1 Comparison of results for Example 1

	Ordinary procedure	Method of this paper	relative Error (%)	Average Error (%)
Sorted eigenvalues	73.322	74.0064	0.933	0.867
	783.711	791.130	0.947	
	886.543	897.446	1.230	
	995.880	995.959	0.008	
	1172.281	1188.475	1.381	
	1258.141	1258.289	0.012	
	1769.001	1803.065	1.925	
	1967.779	1939.171	1.454	
	2006.676	1991.796	0.7415	
	3390.450	3391.636	0.035	

Fig. 1 A 3d near regular structure

Table 2 Comparison of results for Example 2

	Mode 1	Mode 2	Mode 3	matrix size	Average relative error %
Direct method	7.688	13.098	19.564	2184	
$\varepsilon = 10^{-6}$	7.684	13.133	19.564	1348	0.105
$\varepsilon = 10^{-5}$	7.474	13.086	19.171	1343	1.623

frequencies by solving eigenvalue problem for a matrix with size 1348 instead of initial matrix size which was 2148. By another calculation, with accepting average error equals to 1.62 percentage (%) we can find our frequencies by solving eigenvalue problem for a matrix with size 1343 instead of initial matrix size which was 2148.

For this example, for showing how error is reducing by obtaining smaller term "ε" and how matrix dimension is reducing through obtaining larger term "ε", the Fig. 2 is drawn. This graph depicts matrix size for calculation (instead of initial matrix size) against average *error*(%) for first 6 modes of this example through increasing the defined term "ε" from 10^{-5} till 3.2×10^{-5} by step increment 10^{-6}.

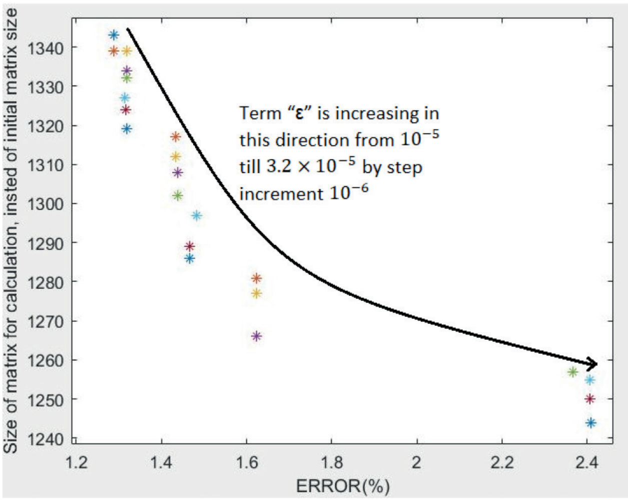

Fig. 2 Variation of error (%) and matrix dimension with respect to the term "ε" for Example 2

7 Example 3

A square steel plate is given in Fig. 3. In this plate there exist chamfer in left and lower corner of the plate. The corresponding regular steel plate is also depicted in Fig. 3. Therefore, we want to use modal information of this regular structure to find frequencies of initial and near regular steel plate. The geometric properties of this structure are as following: Size: 7×7 *in*, thickness: 0.4 *in*, modulus of elasticity: 30×10^6 *psi*, Poisson coefficient: 0.2, density is 0.2836 lb/in^3 (7850 Kg/m^3) and the plate is assumed to be fixed along its four edges.

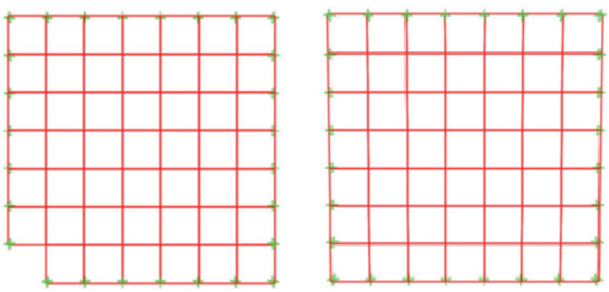

Fig. 3 Structures for Example 3: (A) is near regular steel plate and (B) is corresponding regular steel plate

Stiffness matrix of this structure is calculated by using finite element method. For this analysis, meshes are in square form with size 1×1 *in*, and we used thin shell element. The stiffness matrix and mass matrix of this example are 216×216.

For this plate we want to calculate 20th till 25th frequencies (Note: we calculated square of these numbers) to evaluate accuracy of present paper for calculation of higher modes. Hence, for showing how error is reducing by obtaining smaller term "ε" and how matrix dimension is reducing through obtaining larger term "ε", the Fig. 4 is drawn. This graph depicts matrix size for calculation (instead of initial matrix size) against average *error*(%) for 20th till 25th modes of this example through increasing the defined term "ε" from 0.01 till 0.07 by step increment 0.001.

8 Conclusions

For finding the periods of a structure, one should solve a generalized eigenvalue problem with matrix size equal to the number of degrees of freedom of structure which is a time and storage consuming task. In this paper, modal analysis of the near regular structures is perfomed with using modal information of regular forms. Regular forms are those for which the eigenvalues and eigenvectors can easily be obtained, this nomination also can be referred to structures which associated matrices have less-intricated format or their modal information are already exist. Based

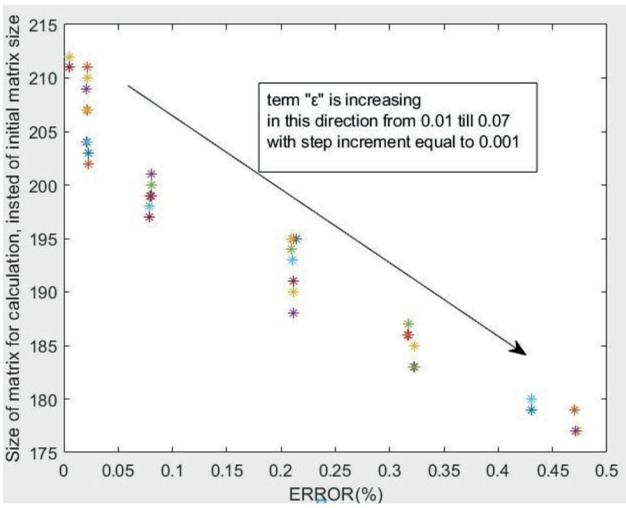

Fig. 4 Variation of error (%) and matrix dimension with respect to the term "ε" for Example 3

on the developed method, by using modal information like frequencies and modal shapes of the corresponding regular structure, it is sufficient to use eigenvalue problem for a matrix with a size that is much smaller than the size of the initial matrix. through this paper a parameter has been utilized as term "ε". By considering this term, error is reduced by obtaining smaller term "ε" and matrix dimension is reduced through obtaining larger term "ε".

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