METHOD FOR THE DYNAMIC ANALYSIS OF NONLINEAR SYSTEMS*

By

E. SIMONYI and M. KASZÁS**

Department for Applied Chemistry, Technical University, Budapest

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Presented by I. PORUBSZKY

The phase-plane methods are among the most used ones for the dynamic analysis of nonlinear systems. The phase-plane methods help to answer the following questions:

1) Has the system steady state or states?

2) Which states are stable and which are unstable ones?

3) Has the system limit cycle or cycles?

4) Are these limit cycles stable, semistable (dependent on the direction namely) or unstable? (Some special phase-plane methods are capable of the semi-quantitative description of the time function of the system [1].)

Now some phase-plane methods, described by the technical literature are examined from the aspect of what is the group of differential equations where they can be used in answering the listed questions.

1) Has the system steady state or states?

Now the task is to resolve the system of equations

$$P(x, y, \alpha_1, \ldots \alpha_n) = 0$$

$$Q(x, y, \alpha_1, \ldots \alpha_n) = 0$$

where

x, y are the variables of the differential equation system,

α_i are the parameters.

If there is no differentiating as to parameters in the equation, the task is to resolve a normal system of equations with two variables. These methods are not to be discussed here.

2) Which steady states of the system are stable and which are unstable ones?

The stability test of simple special points can be done by the first method of LIAPUNOV [2]. This method, however, is suitable for examining but a restricted group of differential equation systems, e.g. a simple system of equations like the following one cannot be treated by it:

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$$P = x^2 + y^2$$

$$Q = x^2 - y^2$$

both Liapunov-determinants equalling zero. Various graphic methods can be utilized at a wider range, of which now the method of isoclines [3] (i.e. that of approaching the trajectories by linear portions) and the method of LIÉNARD [4] are mentioned here. Common advantage of these methods is that they can be widely used; their disadvantage is, however, that their results cannot be generalized, moreover accurate calculations of a single trajectory is a tedious computational and graphical work and many of them are necessary for the total phase-diagram. Another common disadvantage is that not only the numerical value, but the character of the result is highly influenced by the accuracy of the calculations. For example, inaccurate determination of the common point of intersection of several curves can result in several different points of intersection.

3) Has the system limit cycle or cycles?

There are not known necessary and sufficient conditions of the existence of limit cycles, several methods exist, however, that yield sufficient condition of the non-existence of the limit cycles in their certain groups. (E.g. the methods of BENDIXSON, DULAC.) A serious disadvantage of these methods is that they cannot be used in some important cases — e.g. for examining relaxation systems. The limit cycles can be found in some cases by graphical or numerical methods. The disadvantage of these methods is similar as for the determination of steady states: the graphical or numerical errors may distort the character of the system (e.g. they may result in a closed curve instead of a spiral or vice versa). Their common advantage is that they yield the characteristic data of the limit cycle: i.e. its shape, stability, and mostly its frequency as well.

4) Are the limit cycles stable, or unstable?

Several methods are known to determine the stability of the known limit cycles (e.g. the method of KOENIG, the method of the characteristic indices). The advantages and the disadvantages of these methods are more or less similar to the characteristics of the methods mentioned in item 3.

The presented method, developed by us, is suitable for examining systems of nth order. Owing to time shortage and for better understanding the method is described here only for first-order systems.

Before describing the method some notions have to be determined:

a) Directional curve: the geometrical location of points, for which at least one of the time-linear differential equations

$$\frac{dx_i}{dt} = F_i(x_1, \ldots x_n) \equiv F_i\,; \quad i = 1, \ldots n$$

describing the system is equal to zero.

b) Directional vector defined by

$$\overline{W} = \sum_{i=1}^{n} \operatorname{sign}\,[F_i] \cdot \bar{e}_i$$

where $\bar{e}_i$ is the unity vector of direction x_i

The method consists of the following steps:

I. Determination of the directional curves.

II. Determination of the directional vector over the whole phase plane (in case of first order systems over the plane $\dot{x}$, x).

III. The utilization of one sufficient condition of stability. This is as follows:

If the inequality

$$(x_1 - x_{1s})\,P + (x_2 - x_{2s})\,Q < 0$$

is valid over the area of the phase plane containing the point belonging to the steady state, then the point corresponding to the steady state is stable.

In case of first order systems the stability test is done by checking the following condition:

$$\operatorname{sign}\,(x_1 - x_{1s}) = -\operatorname{sign}\; P\,(x_1)$$

Practical application of the condition is shown on the following example:

The temperature changes of the transistor are examined. The change of the heat quantity in unit time is:

$$\frac{dQ}{dt} = \frac{dQ_t}{dt} - \frac{dQ_v}{dt}$$

where Q = the quantity of heat of the transistor
Q_t = the quantity of heat generated in the transistor
Q_v = the quantity of heat dissipated by the transistor.

Here the transistor is supposed to be homogeneous and its temperature to be independent of the place.

The heat, dissipated by the transistor in unit time is the sum of heats dissipated by conduction and radiation:

$$\frac{dQ_v}{dt} = \varepsilon \cdot C_0 \left[\left(\frac{T}{100} \right)^4 - \left(\frac{T_k}{100} \right)^4 \right] + \alpha \cdot F \cdot (T - T_k)$$

where ε = the blackness factor
C_0 = a universal constant
T = the temperature of the surface

T_k = the environmental temperature — in this case the temperature of he housing of the transistor or of the cooling fin
α = the heat-transfer coefficient
F = the entire heat-dissipating surface of the transistor.

The first and the second terms of the equation mean the heat, dissipated by radiation and by conduction, respectively.

The generated heat — in calories — can be expressed as

$$\frac{dQ_t}{dt} = 0.239 \cdot i_c^2 \cdot Z$$

and, as known, the i_c of the transistor is increasing nearly exponentially with the temperature, thus

$$\frac{di_c}{dT} = k \cdot (T - T_k)$$

The rise of the transistor temperature is described by the equation

$$dQ = c \cdot m \cdot dT$$

or, if transposed:

$$c \cdot m \cdot \frac{dT}{dt} = 0.239 \cdot Z \cdot i_c^2 \cdot \exp\left[k\,(T - T_k)\right] -$$

$$- \varepsilon \cdot C_0 \left[\left(\frac{T}{100}\right)^4 - \left(\frac{T_k}{100}\right)^4 \right] - \alpha \cdot F\,(T - T_k)$$

$$\frac{dT}{dt} \equiv P = \overbrace{A \cdot \exp\left[2aT\right]}^{=p_1} \overbrace{- B \cdot T^4 - D \cdot T + C}^{=p_2}$$

Depending on the parameter values, two substantially different modes of operation are possible:

1) If P_1 is invariably greater than P_2, then their curve is as shown in Fig. 1. In this case the heat generation is higher than the heat dissipation for any initial condition, thus the system tends to infinite temperature.

2) If the curves P_1 and P_2 have two intersections, their curves are as shown in Fig. 2; thus the directional curve is as in Fig. 3, then the following statements can be made for the time-dependent operation of the system:

a) If the initial conditions of the system define a temperature below the first point, then the first point is reached as shown in Fig. 4;

b) If the initial temperature is between the two points, then the transient is as shown in Fig. 5;

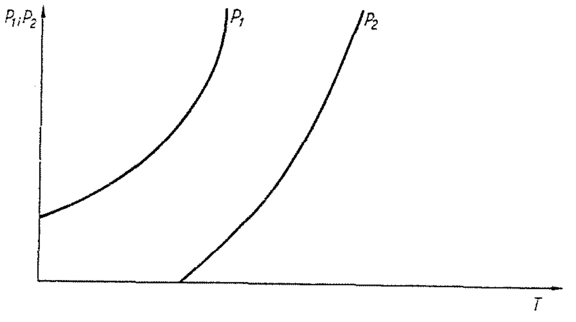

Fig. 1

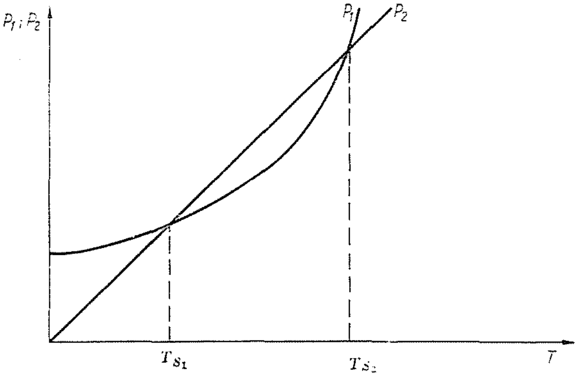

Fig. 2

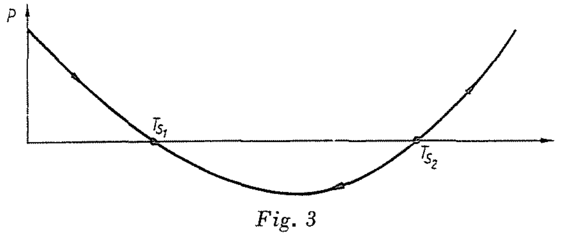

Fig. 3

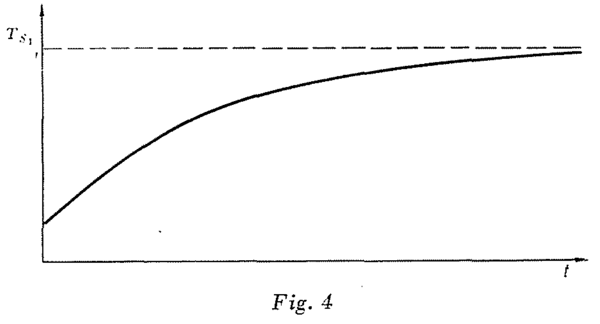

Fig. 4

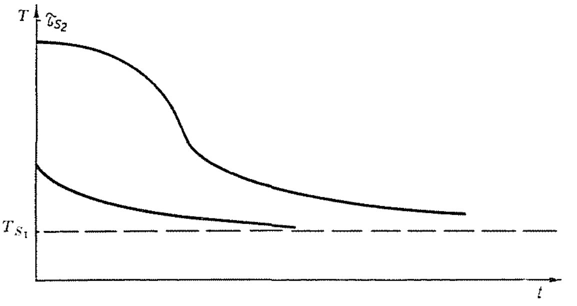

Fig. 5

c) If the initial temperature is higher than the second point, then the system tends to infinite temperature (see Fig. 6).

Most of the examined systems show a relaxational motion. In such systems one parameter varies at a very high (practically infinite) rate (with time), while the other parameter remains practically constant. These two groups of movements will be essentially differently treated, namely, while in first order relaxational systems self-oscillation may occur, in other first order systems this is impossible. Without detailed verifications we assert that self-oscillation in non-relaxation systems is impossible, because self-oscillation is presented by a closed curve on the phase plane (in case of first order systems on the plane $\dot{x}$, x) and this can only be as shown in Figs 7 to 9. (The axis x as a tangent can be considered to intersect, from our aspect.)

$\dot{x}_1$ being positive, x_1 increases to $x_{1_{\max}}$ on the curve of Fig. 7, movement can only be in the direction of increasing x_1 values, thus the curve cannot be "walked around". The situation is similar, but in the opposite direction in Fig. 8.

In Fig. 9, for S_1 and S_2, $\dot{x}_1 = 0$ (corresponding to steady-state condition) which cannot be exceeded without disturbing the system. Thus also here the curve cannot be "walked around".

In relaxation systems the curves close at the "edge" of the phase plane, so closed curves and thereby also self-oscillation may occur.

This statement is illustrated by an example:

Let the movement to be described by the equation:

$$x = \operatorname{tg} t$$

and thus

$$\dot{x} = \frac{1}{\cos^2 t}$$

and

$$\dot{x} = 1 + x^2$$

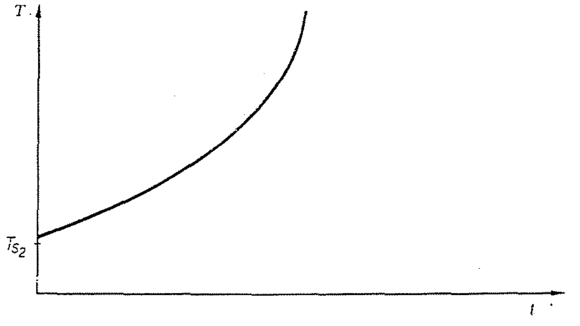

Fig. 6

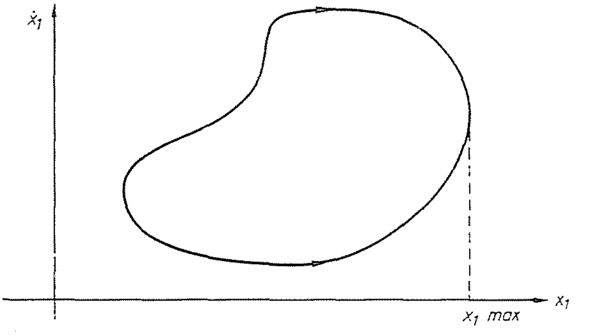

Fig. 7

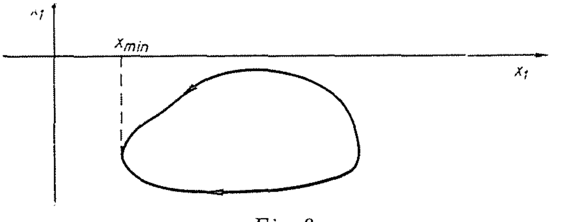

Fig. 8

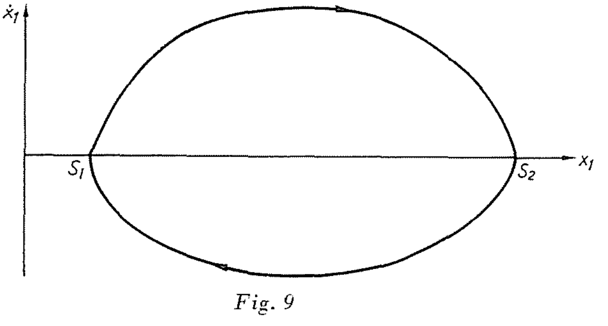

Fig. 9

in all points, except the

$$t = (k+1)\frac{\pi}{2}$$

points of discontinuity. In these points the value of x skips from $+\infty$ to $-\infty$ (Fig. 10) and $\dot{x}$ from $+\infty$ to $-\infty$ and vice versa (Fig. 11). The corresponding phase-picture is shown in Fig. 12, and as a closed curve with finite $T = \pi$ repetition rate.

The method for examining relaxation systems is demonstrated by the generally used tunnel-diode oscillator circuit. As it is known, the current vs. voltage characteristics of the tunnel diode are as shown in Fig. 13.

The equivalent circuit of the simple tunnel-diode oscillator is shown in Fig. 14 where the capacitance of the diode and the stray capacitances of the wiring are neglected. The operation of the circuit is described by the equations:

$$E = U_R + U_L + U_D$$

$$i = f(U_D) = \frac{U_R}{R} = \frac{1}{L}\int U_L \cdot dt$$

$$E = R \cdot f(U_D) = L \cdot \frac{df(U_D)}{dU_D} \cdot \frac{dU_D}{dt} + U_D$$

Let

$$U_D \equiv x > 0\,; \qquad \frac{E}{L} \equiv b > 0$$

$$\frac{R}{L} \equiv d > 0 \qquad \frac{1}{L} \equiv c > 0$$

and thus the directional vector:

$$\dot{x} \equiv P = -a\frac{1}{\frac{df(x)}{dx}} \cdot f(x) - c\frac{1}{\frac{df(x)}{dx}} \cdot x + b \cdot \frac{1}{\frac{df(x)}{dx}}$$

where

$$f'(x) \equiv g(x)$$

is the derivative with respect to voltage of the current vs. voltage curve of the tunnel diode.

The directional curve is analysed by separating the numerator and the denominator: The denominator is shown by the curve in Fig. 15. The numerator is divided into a linear and a nonlinear part (see Fig. 16).

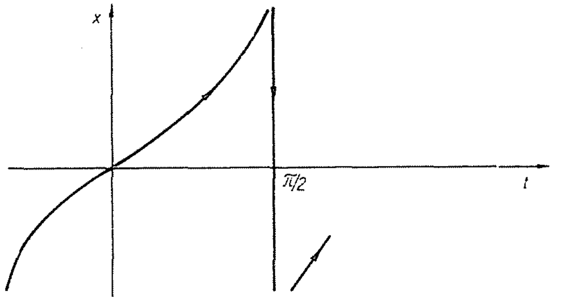

Fig. 10

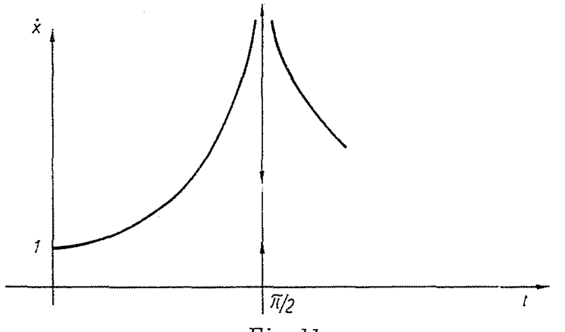

Fig. 11

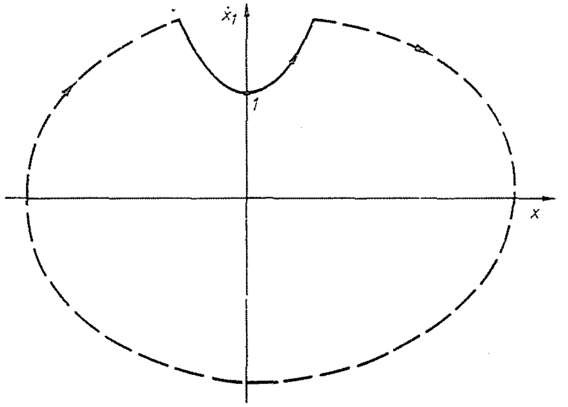

Fig. 12

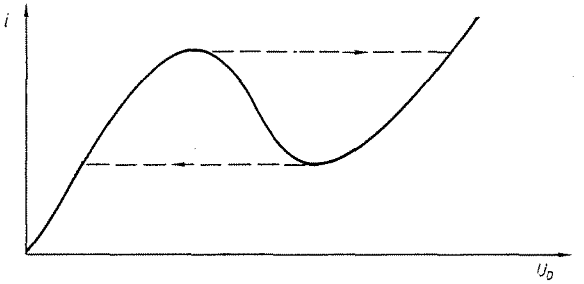

Fig. 13

As the denominator cannot be infinite (on physical reasons), the directional vector is zero, where the numerator is zero, i.e. where the curves for linear and nonlinear parts of the numerator intersect.

The operation of the system is highly dependent on whether there is an intersection on the negative-slope section or not.

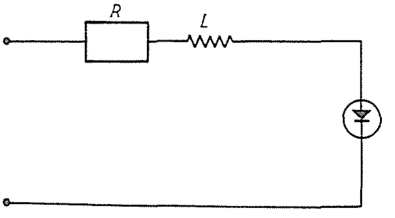

Fig. 14

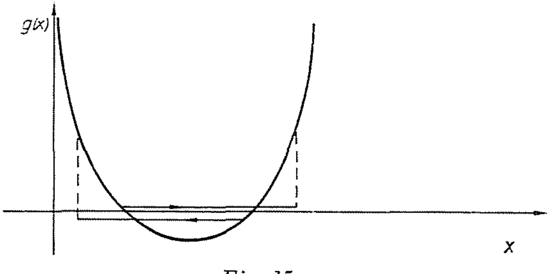

Fig. 15

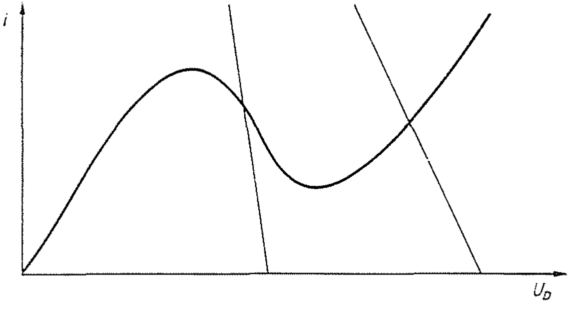

Fig. 16

1) If there is no intersection on the negative-slope section, then the directional curve is as shown in Fig. 17. It appears that in this case the system has a stable operating point, no oscillation can develop, as for any initial condition this point will be reached.

2) If there is an intersection on the negative-slope section, then the directional curve is as shown in Fig. 18. It appears to be a stable limit cycle, self-

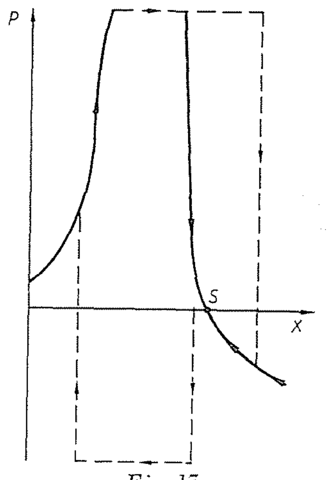

Fig. 17

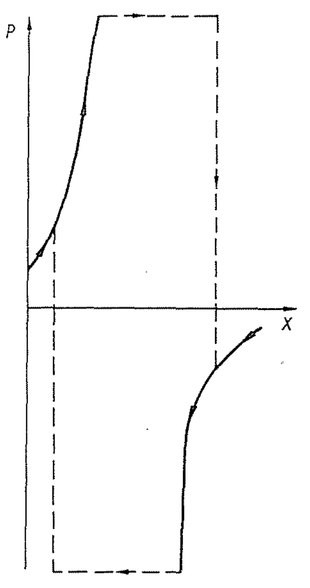

Fig. 18

oscillation arising for any initial condition: the given initial condition may be either on the limit cycle or gets to it. This is conditioned by the solution of equation

$$a \cdot f(x) = b - c \cdot x$$

to meet

$$x_{f(x)\max} < x_{\text{solution}} < x_{f(x)\min} .$$

The resulting time-function is shown by Fig. 19.

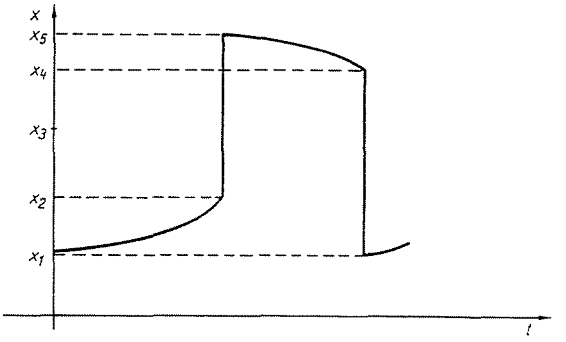

Fig. 19

These examples are likely to prove the described procedure to be an excellent method for examining both practical and theoretical problems.

Summary

A special phase-plane method is described for the dynamic analysis of continuous and relaxation nonlinear systems. This simple and descriptive method suits for designing systems. Usefulness of the method is shown by examples in the field of electronics (thermal stability of transistors; tunnel diode oscillators).

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Endre Simonyi
Miklós Kaszás
Budapest XI., Egry József u. 20, Hungary