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# ON THE PRINCIPLE OF MINIMUM ENTROPY PRODUCTION IN QUASILINEAR CASE AND ITS CONNECTION TO STATISTICAL MECHANICS

Endre KISS

Department of Chemical Physics
Technical University of Budapest
H–1521 Budapest, Hungary

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## Abstract

Studying heat conduction problems in linear and quasilinear ranges for stationary state one can be convinced that the principle of minimum entropy production is valid, but only under special conditions. By using variational calculus we show that the solution of the minimum principle accords totally with that of the energy balance equation for both cases. Of course, the Euler–Lagrange differential equations for linear and quasilinear cases do not give the same solutions and similarly, the temperature distributions differ, too. Nevertheless, according to a deeper analysis we can suspect that only nonlinear heat conduction exists. Investigations from the point of view of the picture representation and a special new method developed for the solution of the variational problem refer to this. The empirical Fourier's law does not seem to fit the energy balance equation because this linear process does not appear exactly in this form in nature. The formal proof for Fourier's law with the energy balance equation very probably is delusive.

*Keywords:* entropy, statistical mechanics.

## 1. Linear Heat Conduction

From the Lagrange densities we can have the solutions of the Euler–Lagrange differential equations for stationary state heat conduction of Fourier type in different representational pictures [1, 2]:

| Fourier picture | Energy picture | Entropy picture |
|---|---|---|
| $\triangle T = 0$ | $\triangle lnT = 0$ | $\triangle \frac{1}{T} = 0$ |

The variational procedure must be carried out according to Gyarmati [1]. Here, in the linear case — on the basis of the equality of the fluxes — in any picture the phenomenological coefficients can be considered to be constant but only independently of each other. The equality of fluxes is:

$$-\lambda \nabla T = -L^{*} \nabla lnT = L \nabla \frac{1}{T} \tag{1}$$

(We use $\lambda$ for the heat conductivity coefficent in Fourier's linear heat conduction law and $\lambda(T)$ for that in the constitutive equation of quasilinear heat conduction.) But if in a representational picture we consider the phenomenological coefficient to be constant then from the point of view of the so-yielded differential equation and its picture, the differential equation elaborated in the same way gives us in another picture a quasilinear solution with temperature dependent phenomenological coefficient.

## 2. Quasilinear Heat Conduction

According to our prediction some quasilinear Euler–Lagrange equation for the Fourier picture appears in the linear Euler–Lagrange equation of the energy and the entropy pictures, respectively:

Energy picture:

$$\triangle lnT = \nabla(\nabla lnT) = \nabla\left(\frac{\nabla T}{T}\right) = (\nabla T)^2 - T\triangle T = 0; \tag{2}$$

Entropy picture:

$$\triangle\frac{1}{T} = \nabla(\nabla 1T) = \nabla(-\frac{\nabla T}{T^2}) = 2(\nabla T)^2 - T\triangle T = 0. \tag{3}$$

For to the verification of these quasilinear Euler–Lagrange equations we write the quasilinear differential equation with a temperature dependent phenomenological coefficient for heat conduction in Fourier picture [2]:

$$\rho_0 c_v(T)\frac{\partial T}{\partial t} - \nabla\cdot[\lambda(T)\nabla T] = 0, \tag{4}$$

but more explicitly,

$$\rho_0 c_v(T)\frac{\partial T}{\partial t} - \lambda'(T)(\nabla T)^2 - \lambda(T)\triangle T = 0. \tag{5}$$

where

$$\lambda'(T) = \frac{\partial\lambda}{\partial T}. \tag{6}$$

*Eq.* (5) is a quasilinear partial differential equation. For the case of stationary state we can write,

$$\lambda'(T)(\nabla T)^2 + \lambda(T)\triangle T = 0. \tag{7}$$

From the equality of the fluxes now in the quasilinear case we get that instead of $-\lambda\nabla T$ we have to write $-\lambda(T)\nabla T$ in the Fourier picture for the flux so that constants $L^*$ and $L$ will be

$$L^* = T\lambda(T) \qquad \text{and} \qquad L = T^2\lambda(T), \tag{8}$$

and using (8) we can compare equation (7) with equations (2) and (3). Therefore we can write in the energy picture that

$$L^*(\nabla T)^2 - L^* T \triangle T = 0, \tag{9}$$

or in another form

$$\frac{\lambda(T)}{T}(\nabla T)^2 - \lambda(T)\nabla T = 0. \tag{10}$$

From the comparison we can get for $\lambda(T)$ and $\lambda'(T)$:

$$\lambda(T) = \frac{C_2}{T} \qquad \text{and} \qquad \lambda'(T) = -\frac{C_2}{T^2}, \tag{11}$$

which coincide totally with the results of statistical mechanics according to the $1/T$ ratio in the high temperature range over 100 K [3, 4, 5]. It is known from the literature that the value of $C_2$, e.g., for germanium is about 20000 W/m. On the other hand, in the entropy picture we get from the comparison that

$$2L(\nabla T)^2 - LT\nabla T = 0, \tag{12}$$

and in another form

$$\frac{2\lambda(T)}{T}(\nabla T)^2 - \lambda(T)\triangle T = 0. \tag{13}$$

Now we get for $\lambda(T)$ and $\lambda'(T)$:

$$\lambda(T) = \frac{C_1}{T} \qquad \text{and} \qquad \lambda'(T) = -\frac{2C_1}{T^3}, \tag{14}$$

The $\lambda(T)$ values resulting from expressions (11) and (14) are the same but with different $C_2$ and $C_1$ constants, where $C_1 = TC_2$. From the point of view of the temperature distributions in stationary state we consider a one-dimensional heat conduction problem, where heat is conducted through a plate. The thickness of the plate is $l$ and the surfaces are kept at constant temperatures. The boundary conditions are

$$T(x = 0) = T_0 \qquad \text{and} \qquad T(x = l) = T_1,$$

then the temperature distribution is

$$T(x) = T_0(T_1/T_0)^{x/l} \qquad \text{in the energy picture,} \tag{15}$$

and

$$T(x) = \frac{1}{\frac{1}{T_0} - \left(\frac{T_1 - T_0}{T_1 T_0}\right)\frac{x}{l}} \qquad \text{in the entropy picture,} \tag{16}$$

which are the solution of the quasilinear Euler–Lagrange equations for the Fourier picture with quasilinear phenomenological coefficients (11,13).

Reminder: the temperature distribution with stationary-state heat conduction of constant phenomenological coefficient – i.e., Fourier's case differential equation of linear type in the Fourier picture – is

$$\triangle T = 0, \qquad T(x) = T_0 + (T_1 - T_0)\frac{x}{l}, \tag{17}$$

which is the formal linear solution for the Fourier picture with linear phenomenological coefficient.

### 3. Direct Variation for the Quasilinear Case

From the equality of the fluxes with the expression

$$-\lambda(T)\nabla T = -L^* \nabla lnT = L\nabla \frac{1}{T}, \tag{18}$$

we write for quasilinear case the Lagrange densities in different pictures.

*Table 1.*

| | |
|---|---|
| $\lambda(T)\nabla T\nabla T = \mathcal{L}_T(T) = T^2\sigma$, | Fourier picture, |
| $\lambda(T)\nabla T\frac{\nabla T}{T} = \mathcal{L}_{lnT} = T\sigma = L^*\frac{(\nabla T)^2}{T^2}$, | energy picture, |
| $\lambda(T)\nabla T\frac{\nabla T}{T^2} = \mathcal{L}_{1/T} = \sigma = L\frac{(\nabla T)^2}{T^4}$, | entropy picture, |

Now we take the Euler–Lagrange differential equation in the Fourier picture according to Gyarmati [1], but for $\mathcal{L}_T$ we substitute $\mathcal{L}_{lnT}$ or $\mathcal{L}_{1/T}$ in order to extend the range of the admissible functions as a recency, then after having solved the variation problem we can have the quasilinear differential equations (2) and (3) again. It is due to the connections between the Lagrange densities, which come from the equality of the fluxes. The Lagrange density according to the Fourier picture shows the existence of nonlinear heat conduction in stationary state but not that of the linear one. Generally we can write for the system of Lagrange densities the connections between linearity and quasilinearity in the different pictures, i.e.,

So it can be seen that the principle of minimum entropy production is valid for some quasilinear equation, too. As to the energy balance equation we can show the formal proof in the linear case

$$\nabla \cdot [\lambda \nabla T] = \triangle T = 0, \tag{19}$$

in the quasilinear case the real form is

$$\nabla \cdot [\lambda(T)\nabla T] = \lambda'(T)(\nabla T)^2 + \lambda(T)\nabla T = 0, \tag{20}$$

*Table 2.*

| | |
|---|---|
| $\mathcal{L}_T =$ | $T^2\mathcal{L}_{1/T}(T) = T\mathcal{L}_{lnT}(T)$, |
| $\mathcal{L}_{lnT} =$ | $T\mathcal{L}_{1/T}(T) = T^{-1}\mathcal{L}_T(T)$, |
| $\mathcal{L}_{1/T} =$ | $T^{-1}\mathcal{L}_{lnT}(T) = T^{-2}\mathcal{L}_T(T)$. |

therefore, the Euler–Lagrange equations of the principle of minimum entropy production and the energy balance equation coincide totally in the quasilinear range.

## 4. Quasilinear Solutions for $L(t)$ and $L^*(t)$

In the case of $L(T)$ the differential equations are

$$2\lambda(\nabla T) + \lambda T \triangle T = 0 \quad \text{in the Fourier picture,} \tag{21}$$

$$2L^*(\nabla T)^2 + L^* T \triangle T = 0 \quad \text{in the energy picture.} \tag{22}$$

In case of $L^*(T)$ the differential equations are

$$\lambda(\nabla T)^2 + \lambda T \triangle T = 0 \quad \text{in the Fourier picture,} \tag{23}$$

$$3L(\nabla T)^2 + LT \triangle T = 0 \quad \text{in the entropy picture.} \tag{24}$$

The further procedure is the same as was in the case of (9) and (12), i.e., we express the differential equations with the aid of $L(T)$ and $L^*(T)$ on the basis of the equality of the fluxes, respectively. The temperature distributions to differential equations (21–24) are as follows:

$$T(x) = \sqrt[3]{(T_1^3 - T_0^3)\frac{x}{l} + T_0^3},$$

$$T(x) = \sqrt[3]{(T_1^3 - T_0^3)\frac{x}{l} + T_0^3},$$

$$T(x) = \sqrt{(T_1^2 - T_0^2)\frac{x}{l} + T_0^2},$$

$$T(x) = \sqrt{\frac{-1}{\left(\frac{2T_1^2 - T_0^2}{T_1^2 T_0^2}\right)\frac{x}{l} - \frac{2}{T_0^2}}}.$$

The boundary conditions for these solutions are as earlier.

## 5. Summary

For stationary state heat conduction there are three differential equations for the linear case according to the three representational pictures. But this is an old view because we saw that Fourier's law in the linear case ($\triangle T = 0$) does not exist in nature so there are only two right forms for linear heat conduction in the energy and entropy pictures. In the quasilinear case we can write altogether six differential equations, i.e., in each representational picture one can write the quasilinear differential equations of the two other pictures. Here, in accordance with these facts, we can write only two nonlinear differential equations for $\lambda(T)$ in the energy and in the entropy pictures. The other four differential equations as solutions are only formal from our point of view. At the variational disposal we used Gyarmati's solution but with a principle of recency for the admissible function in the quasilinear case. Here, we emphasize that at the variational problem the time derivative must be frozen. In a stationary-state heat conduction problem this is a very important requirement for the variational procedure. As to the linear and quasilinear solutions, from time to time there are misunderstandings in the literature, frequently in connection with the principle of minimum entropy production. So, e.g., in [8] the linear and quasilinear solutions for stationary state heat conduction could not be distinguished and for this reason was the principle of the minimum entropy production criticized. The proper use of the representational pictures gives a clear sight and it seems that the principle of the minimum entropy production [9, 10] operates well in the linear and quasilinear ranges of heat conduction problems, too. The temperature dependence of the phenomenological coefficients, i.e., the heat conduction coefficients show a good correlation with the results of the solid state physics and of statistical mechanics [11, 12]. The variational method is a good tool in the solutions of heat transfer problems [1, 6]. It seems that heat conduction is theoretically a nonlinear process [13, 7]. Therefore remember that if you are smoking you can blow a ring of smoke but not a half one. Realistic problems encountered in engineering applications are nonlinear. So is the situation with Fourier's law in the nature, because an experimental law must be placed in a whole theoretical frame. *Quidquid agis prudenter agas et respice finem.*

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