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THERMOMECHANICS OF HETEROGENEOUS MATERIALS WITH WEAKLY NONLOCAL MICROSTRUCTURE

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Abstract

A unified thermomechanical framework is presented for the theory of mechanically elastic materials the physical description of which requires the consideration of additional variables of state and of their gradients (weak nonlocality). This includes both the case of additional degrees of freedom carrying their own inertia and the case of diffusive internal variables of state. In view of practical applications to fracture and propagation of phase-transition fronts, special attention is paid to the construction and immediate consequences of the equations of balance of canonical momentum (on the material manifold) and energy at regular points and at jump discontinuities. In particular, the general expression of the dissipation at, and of the driving force acting on, phase-transition fronts is formally obtained in such a broad framework. Brief applications include thermoelastic conductors (e.g., shape-memory alloys) and elastic ferromagnets in which both spin inertia and ferromagnetic exchange forces (magnetic ordering) are taken into account.

Keywords: thermodynamics, inhomogeneity, nonlocality, microstructure.

1. Introduction

With a tremendous development in both the conception and uses of composite materials, smart materials exhibiting phase transitions, and materials exploiting combined physico-mechanical effects, it has become a necessity to think in more rational terms about a somewhat general framework. This is what we called, perhaps inappropriately, the theory of *material inhomogeneities* once it was recognized that many of the relevant effects are manifested in some characteristic way in the *canonical balance laws* of continuum physics, namely, the balance of canonical (material) momentum and of energy at both regular and singular material points (MAUGIN [1], [2]). Allied to a formulation based on the *principle of virtual power* (e.g., MAUGIN [3]) and the powerful notions of *evolving microstructure* and *internal variables*

of state (MAUGIN and MUSCHIK [4]), this indeed provides a satisfactory general framework into which such macroscopically irreversible phenomena as fracture and the progress of phase-transition fronts fit naturally.

In the present contribution, in order to exemplify our viewpoint, we shall consider a sufficiently large class of materials in which, say, the solid-like material exhibits some elasticity but its complete physical description requires considering one (or several) additional variable(s) (5) which allow(s) a *weak nonlocality* (cf., MAUGIN [5]) in the sense that its (their) gradient(s) need(s) also to be involved in some way. We shall first formally develop the scheme, specifying only at a later stage if this additional variable presents a classical inertia (i.e., it is an additional internal degree of freedom *per se*) or it shares most of the properties associated with so-called *internal variables of state* (cf., Ref. [4]). Special emphasis is placed on obtaining the expression of *canonical* (material) balance laws as they play a fundamental role in designing criteria of the progress of defects. We remind the reader that material inhomogeneities in a broad sense – including dislocations, growing cavities, cracks, phase transitions – are viewed as 'defects' of various dimensionalities in this theory.

A short background is given in Section 2. Canonical balances of momentum and energy are expanded in Section 3. This decides the expression of the second law of thermodynamics at regular material points. The associated jump relations are developed in Section 4. We briefly deal with some illustrative examples in Section 5. A more extended work on the subject will be published later on (MAUGIN [6]).

2. Background

Using standard notations of continuum mechanics (e.g., TRUESDELL and NOLL [7], MARSDEN and HUGHES [8]) let B, an open simply connected subset of the material manifold $\mathcal{M}^3$, be our material body of 'particles' $\mathbf{X}$ and $\rho_0(\mathbf{X})$ the mass measure at the reference configuration $\mathcal{K}_R$. The actual placement of $\mathbf{X}$ at time t is given by the (supposedly) smooth function $\mathbf{x} = \chi(\mathbf{X}, \mathbf{t})$. In the absence of body forces and for an *essentially* elastic (hyperelastic) material, at each regular point $\mathbf{X}$ we have the following two basic local laws in the *Piola-Kirchhoff* form:

- *balance of mass*

$$\partial \rho_0 / \partial t|_{\mathbf{x}} = 0 \tag{1}$$

- *balance of linear momentum*

$$\partial \mathbf{p} / \partial t|_{\mathbf{x}} - \operatorname{div}_R \mathbf{T} = 0 \tag{2}$$

These are *strict* conservation laws. Here we have set

$$\mathbf{p} = \rho_0(\mathbf{X})\mathbf{v}, \mathbf{T} = \partial W / \partial \mathbf{F} \ , \tag{3}$$

where

$$\mathbf{v} := \partial\chi/\partial t|_{\mathbf{X}} \ , \quad \mathbf{F} := \partial\chi/\partial\mathbf{X}|_{\mathbf{t}} \equiv \nabla_{\mathbf{R}}\chi. \tag{4}$$

We assume $J_F = \det \mathbf{F} > 0$ always $(\forall t)$ so that we also have

$$\mathbf{V} := \partial\chi^{-1}/\partial t|_{\mathbf{x}}, \quad \mathbf{F}^{-1} := \partial\chi^{-1}/\partial\mathbf{x}|_t \equiv \nabla\chi^{-1} \tag{5}$$

and we can check that

$$\mathbf{V} = -\mathbf{F}^{-1}\cdot\mathbf{v} \ , \qquad \mathbf{v} = -\mathbf{F}\cdot\mathbf{V} \ , \tag{6}$$

$$\mathbf{F}^{-1}\cdot\mathbf{F} = \mathbf{1}_R \ , \qquad \mathbf{F}\cdot\mathbf{F}^{-1} = \mathbf{1} \ , \tag{7}$$

where $\mathbf{1_R}$ and $\mathbf{1}$ are unit dyadics at $\mathcal{K}_R$ and $\mathcal{K}_t$ (actual configuration), respectively. Of some use can be the following common measures of finite strain (T = transpose)

$$\mathbf{C} = \mathbf{F}^T\cdot\mathbf{F}, \quad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1}_R) \ . \tag{8}$$

We need to say some words on the scalar-valued function W present in the definition $(3)_2$. It is for sure an *energy* per unit volume of the material at K_R. Henceforth we shall assume that it admits the following functional dependence

$$W = \overline{W}(\mathbf{F}, \alpha, \nabla_R\alpha, \theta; \mathbf{X}) \ , \tag{9}$$

where θ – such that $\theta > 0$, $\inf\theta = 0$ – is called the thermodynamical temperature, and α is any suitable field variable $\alpha(\mathbf{X}, t)$ for our physical description. Hence, W is a *free energy*. The fact that $\nabla_R\alpha$ appears in (9) implies a *weak nonlocality* as regards the physical property related to α – hence a characteristic length $l \sim |\alpha|/|\nabla_R\alpha|$; while the explicit appearance of $\mathbf{X}$ indicates a possible, here smooth, material inhomogeneity. It is clear that (2) and (9) incorporate an extremely wide class of materials, thermoelastic ones being among the simplest. The consideration of higher gradients of $\chi(\mathbf{x}, t)$ than the first would complicate the whole picture. As no memory effect is included, the local statement of the balance of *moment of momentum* is equivalent to the invariance under $SO(3)$ in $\mathcal{K}_t$ of the scalar-valued function W. This can be precisely written down only with a specification of the tensorial nature of α. Formally, however, we can write the corresponding condition as

$$(\mathbf{F}\cdot\mathbf{T} + SO(W|\alpha))_A = 0 \ , \tag{10}$$

where subscript A denotes the operation of skewsymmetrization and $SO(W|\alpha)$ indicates the action of an infinitesimal generator of $SO(3)$ on W with respect to its α functional dependence only. In the absence of α, (10) reduces to the classical condition $\mathbf{F}\cdot\mathbf{T} = \mathbf{T}^T\cdot\mathbf{F}^T$, which represents the symmetry of the Cauchy stress tensor or the symmetry of the first Piola stress $\mathbf{T}$ 'with respect to $\mathbf{F}$'.

3. Canonical Balance Equations

We assume that W is concave in θ and no convexity conditions insofar as $\mathbf{F}$ and α are concerned, and set

$$S := -\partial\overline{W}/\partial\theta, \quad \overline{\mathcal{A}} := -\partial\overline{W}/\partial\alpha, \quad \mathcal{M} := \partial\overline{W}/\partial(\nabla_R\alpha)\,, \tag{11}$$

$$\mathcal{A} := \overline{\mathcal{A}} + \nabla_R \cdot \mathcal{M} = -\delta\overline{W}/\delta\alpha\,, \tag{12}$$

where $\delta/\delta\alpha$ denotes the Euler–Lagrange functional derivative with respect to the components of α. By taking the inner product of (2) on the right by either $\mathbf{F}$ or $\mathbf{v}$ and integrating by parts on account of *Eqs.* (3), (1), (9), (11) and (12), we arrive at the following result:

- *Theorem 1.* At each regular material point $\mathbf{X} \in \mathcal{B}$, at time t, the following *canonical* covectorial and scalar balance laws hold:
- *balance of canonical momentum:*

$$\left.\frac{\partial\mathcal{P}}{\partial t}\right|_{\mathbf{x}} - \left\{ \mathrm{div}_R\mathbf{b} + \mathbf{f}^{inh} + \mathbf{f}^{th} \right\} = \mathcal{A}\nabla_R\alpha \tag{13}$$

- *balance of energy:*

$$\left.\frac{\partial\mathcal{H}}{\partial t}\right|_{\mathbf{x}} - \left\{ \nabla_R \cdot \mathbf{G} + q^{inh} + q^{th} \right\} = -\mathcal{A}\dot{\alpha}\,, \tag{14}$$

where

$$\dot{\alpha} = \left.\frac{\partial\alpha}{\partial t}\right|_{\mathbf{x}}\,, \tag{15}$$

$$\mathcal{P} = -\mathbf{F}^T \cdot \mathbf{p} = \rho_0\mathbf{C}\cdot\mathbf{V}\,, \tag{16}$$

$$\mathbf{b} = -\left(\mathcal{L}^{th}\mathbf{1}_R + \mathbf{T}\cdot\mathbf{F} + \mathcal{M}\otimes(\nabla_R\alpha)^T\right)\,, \tag{17}$$

$$\mathbf{G} = \mathbf{T}\cdot\mathbf{v} + \mathcal{M}\cdot\dot{\alpha}\,, \tag{18}$$

$$\mathbf{f}^{th} = S\nabla_R\theta, \quad q^{th} = \theta\left.\frac{\partial S}{\partial t}\right|_{\mathbf{x}}\,, \tag{19}$$

$$\mathcal{H} = E + K = S\theta + \mathcal{P}\cdot\mathbf{V} - \mathcal{L}^{th}\,, \tag{20}$$

$$K = \frac{1}{2}\rho_0(\mathbf{X})\mathbf{v}^2 = \frac{1}{2}\rho_0(\mathbf{X})\mathbf{V}\cdot\mathbf{C}\cdot\mathbf{V}\,, \tag{21}$$

$$E = S\theta + \overline{W}, \tag{22}$$

$$\mathcal{L}^{th} = K - \overline{W} = \overline{\mathcal{L}}^{th}(\mathbf{v}, \mathbf{F}, \alpha, \nabla_R\alpha, \theta; \mathbf{X}), \tag{23}$$

and

$$\mathbf{f}^{inh} = \left.\frac{\partial\overline{\mathcal{L}}^{th}}{\partial\mathbf{X}}\right|_{expl}, \quad q^{inh} = \left.\frac{\partial\overline{\mathcal{L}}^{th}}{\partial t}\right|_{expl} \equiv 0. \tag{24}$$

Here E is the *internal* energy per unit volume at $\mathcal{K}_R$ and $\mathcal{L}^{th}$ and $\mathcal{H}$ may be thought as *effective Lagrangian* and *Hamiltonian* densities in the presence of *dissipative* effects. The notation in *Eqs.* (24) means that all fields are kept constant in the computation of these partial derivatives. As a matter of fact, with $q^{inh} \equiv 0$, local energy conservation (first law of thermodynamics) requires that *Eq.* (14) should be a *strict* conservation law, from which there follows that the quantity $q^{th} - \mathcal{A}\dot{\alpha}$ must be a material divergence. Thus (14) takes the form

$$\left.\frac{\partial \mathcal{H}}{\partial t}\right|_{\mathbf{x}} - \nabla_R(\mathbf{T}\cdot\mathbf{v} + \mathcal{M}\cdot\dot{\alpha} - \mathbf{Q}) = 0 \tag{25}$$

while we have simultaneously the local *entropy balance* as

$$\left.\frac{\partial S}{\partial t}\right|_{\mathbf{x}} + \nabla_R\cdot\mathbf{S} = \sigma^{th} + \sigma^{intr}, \tag{26}$$

where we have set

$$\mathbf{S} = \mathbf{Q}/\theta, \quad \sigma^{th} = -\mathbf{S}\cdot\nabla_R(\ln\theta), \quad \sigma^{intr} = \theta^{-1}(\mathcal{A}\dot{\alpha}). \tag{27}$$

Following BRIDGMAN [9], the second law of thermodynamics at regular material points requires that the right-hand side in (26) should be non-negative:

$$\sigma_{\mathbf{B}} = \sigma^{th} + \sigma^{intr} \geq 0\,, \tag{28}$$

in which σ^{intr} stands for the bulk *intrinsic* entropy source, and $\mathbf{Q}$ and $\mathbf{S}$ are the heat-flux and entropy-flux vectors. Other possible consistent choices of $\mathbf{S}$ and σ^{intr} are discussed by MAUGIN [6]. The present choice suits well the spirit of the theory of *internal variables of state* if the latter's definition is relaxed so as to allow the possible occurrence of *diffusion* (via $\nabla_R\alpha$) and *controllability* at a boundary (cf. MAUGIN [10]). Indeed, a naïve implementation of the classical theory of irreversible processes (TIP) (DE GROOT and MAZUR [11]) to the term σ^{intr} considered separately from σ^{th} would yield a generally *non-linear evolution-diffusion equation* for α hence with coupling to the 'main' field equation (2) an extraordinarily rich complex of nonlinear dynamical behaviours, of which the simplest one is exemplified by the so-called *intertial-dissipative* systems, exhibited as demonstrated by ENGELBRECHT and MAUGIN [12] for thermoelastic conductors. Notice that a more singular relationship between $\mathcal{A}$ and $\dot{\alpha}$ can be envisaged within the framework of convex analysis such as a 'plastic type' of evolution

$$\dot{\alpha} \in N_C(\mathcal{A}), \quad \mathcal{A} = -\delta\overline{W}/\delta\alpha, \tag{29}$$

where N_C denotes the 'cone of outward normals' to a convex set C in $\mathcal{A}$ space (compare to elastoplasticity in MAUGIN [13]). An interesting feature here is that only *one* generalized conjugate force $\mathcal{A}$ intervenes although we introduced gradients of α. This, in our opinion, is much more preferable than the proposals of some authors (e.g., AIFANTIS [14]).

4. Jump Relations

The important classes of singular surfaces exhibiting jump discontinuities in the field and/or derivatives of the fields are *shock waves* and *phase-transition fronts*. The first class, although implied by dissipation, see the *structure* of shock waves, is based on the consideration of jump relations associated with a strictly conservative system and takes internal energy (function of entropy) as the primary energetic ingredient. The second class involves generally dissipation and builds essentially on the notion of free energy (Landau) so that in our materially-oriented mind, we shall consider the second class of greater interest. Henceforth we focus on this class in an even more restricted way by considering so-called *coherent* phase-transition fronts at which two allotropic phases of the material co-exist at a single temperature $\overline{\theta}$. This last condition imposes that the jump of θ, $[\theta]$, be zero at the phase-transition front Σ, a sufficiently regular surface with unit normal $\mathbf{N}$ (from phase− to phase+) uniquely defined at all of its points and material velocity $\overline{\mathbf{V}}$. It was shown in previous works (MAUGIN and TRIMARCO [15], TRIMARCO and MAUGIN [16]) that the *coherency condition* (no defect such as dislocations along Σ or continuity of lattice sites at Σ) in time, reads as follows on the material manifold ($[f] = f^+ - f^-$):

$$[\mathbf{V}] = 0\,. \tag{30}$$

It is also shown that

$$\overline{V}_N \equiv (\overline{\mathbf{V}} \cdot \mathbf{N}) = V_N^+ = V_N^-\,. \tag{31}$$

The problem consists now in writing down the jump relations at Σ, which are associated to the balance laws (1), (2), (13), (25) and (26).This is solved by applying the formalism of *weak solutions* according to which (see, e.g., BAZER and ERICSON [17]) we can formally replace the partial differential operators ∇_R and $\partial/\partial t|_{\mathbf{X}}$ applied to a function $f(\mathbf{X}, t)$ by the operators $\mathbf{N}\cdot[.]$ and $-\overline{V}_N[.]$, respectively, and introduce as yet unknown source terms in those jump relations that correspond to *non*-strict conservation laws. Applying this strategy we have at once the following jump relations at Σ

$$\overline{\mathbf{V}}_N[\rho_0] = 0\,, \tag{32}$$

$$\mathbf{N}\cdot[\mathbf{T} + \overline{\mathbf{V}} \otimes \mathbf{p}] = 0\,, \tag{33}$$

$$\mathbf{N}\cdot[\mathbf{b} + \overline{\mathbf{V}} \otimes \mathcal{P}] + \mathbf{f}_\Sigma = 0\,, \tag{34}$$

$$\mathbf{N}\cdot[\mathcal{H}\overline{\mathbf{V}} + \mathbf{T}\cdot\mathbf{v} + \mathcal{M}\dot{\alpha} - \mathbf{Q}] = 0\,, \tag{35}$$

$$\mathbf{N}\cdot[S\overline{\mathbf{V}} - \mathbf{S}] = \sigma_\Sigma \geq 0\,, \tag{36}$$

while the heat equation (*Eq.* (26) multiplied by θ) yields

$$\mathbf{N}\cdot[\theta S\overline{\mathbf{V}} - \mathbf{Q}] - q_\Sigma = 0\,. \tag{37}$$

We have accounted for the fact that $[\theta] = 0$ across Σ. *Eqs.* (32), (33), (35) and (36) are 'classical' jump relations. *Eqs.* (34) and (37) are additional relations which must be consistent with the first set. It is this consistency condition that will yield useful information. First, consistency of (37) with (36) leads to

$$q_\Sigma \geq 0 . \tag{38}$$

Secondly, we know the formal expressions of $\mathbf{T}$, $\mathbf{p}$, $\mathbf{b}$, $\mathcal{P}$, $\mathcal{H}$, $\mathcal{M}$ and $\mathbf{S}$. It remains to find, if possible, the expression of $\mathbf{f}_\Sigma$. For this purpose we evaluate the power expanded by $\mathbf{f}_\Sigma$ in a motion of Σ, that is we compute

$$P(\mathbf{f}_\Sigma) = \mathbf{f}_\Sigma \cdot \overline{\mathbf{V}} . \tag{39}$$

Without making any assumptions as regards the time evolution of α, we let the reader prove first that

$$q_\Sigma = P_\Sigma - \left[\mathbf{N} \cdot \mathcal{M} \left. \frac{\partial \alpha}{\partial t} \right|_{\mathbf{x}} \right] , \tag{40}$$

where the time derivative is taken fixed at $\mathbf{x}$ (i.e., an Eulerian time derivative) and further that the following result holds:

Theorem 2. At points $\mathbf{X} \in \Sigma$ the following thermodynamic restriction holds:

$$q_\Sigma = -\overline{\mathbf{V}}_{\mathbf{N}} \mathcal{HUGO} - \left([\mathbf{N} \cdot \mathcal{M}] \left\langle \left. \frac{\partial \alpha}{\partial t} \right|_{\mathbf{X}} \right\rangle + \langle \mathbf{N} \cdot \mathcal{M} \rangle \left[\left. \frac{\partial \alpha}{\partial t} \right|_{\mathbf{x}} \right] \right) \geq 0 , \tag{41}$$

where the scalar functional $\mathcal{HUGO}$ $(S^\pm)$ reads

$$\mathcal{HUGO}(S^\pm) = \left[W - \langle \mathbf{N} \cdot \mathbf{T} \rangle \frac{\partial \chi}{\partial \mathbf{N}} - \langle \mathbf{N} \cdot \mathcal{M} \rangle \frac{\partial \alpha}{\partial \mathbf{N}} \right] , \tag{42}$$

in which $\partial / \partial N = \mathbf{N} \cdot \nabla_R$ denotes the normal derivative and $\langle f \rangle = \frac{1}{2}(f^+ + f^-)$ denotes the average value at Σ. The proof of the theorem is somewhat perilous and will be given in full in another paper [6]. Notice, that the proof does not require any hypothesis of quasi-staticity so that, indeed, it is always the *free* energy W which appears in $\mathcal{HUGO}$, and that in agreement with original thoughts of J.W. Gibbs and P. Duhem, kinetic energy does not contribute to the local structural rearrangement accompanying the phase transition at Σ. This result is here *proven* as kinetic energy and inertia were present in the other equations (33), (34) and (35). But it is *Eq.* (41) which governs the *irreversible progress* of Σ. Expression (41) would not be very useful if it remained of this general form. But it happens that the quantity within large parentheses vanishes identically in *all* known cases. Indeed, either α is of the internal variable of state type or it is an additional

internal degree of freedom carrying its own inertia. In the first case, the conditions

$$[\mathrm{N}\cdot\mathcal{M}]=0,\qquad \left[\left.\frac{\partial\alpha}{\partial t}\right|_{\mathrm{X}}\right]=0 \tag{43}$$

are, respectively, the natural jump relation associated with the non-linear evolution-diffusion equation for α, and the condition of *coherency*, for the variable α, at Σ. In the second case $(43)_2$ still is the admitted additional condition of coherency and $[\mathrm{N}\cdot\mathcal{M}]$ is expressible in terms of the jump of the inertia associated with α. This, multiplied by $\langle\partial\alpha/\partial t|_x\rangle$, would render zero after combination with a kinetic-energy term and a flux of canonical momentum which would have appeared is (35) and (34), had we envisaged this possibility from the start. Hence, in *all* cases we have the following important result for *coherent* phase-transition fronts:

$$\overline{\theta}\sigma_\Sigma=q_\Sigma=f_\Sigma\overline{V}_N\geq 0\,, \tag{44}$$

and

$$f_\Sigma+\mathcal{HUGO}=0\quad\text{at}\quad\Sigma\,. \tag{45}$$

The latter is a *surface balance equation* involving the *scalar driving force* f_Σ, which is restrained, and therefore determined, by the local dissipation inequality (44), and the *Hugoniot-Gibbs functional* $\mathcal{HUGO}$ which depends on the field solution on both sides of Σ. Clearly: (i) *Eq.* (44) refers to a normal growth of one phase in the other (this normality is related to the absence of defects at Σ) and (ii) this equation can be used to devise an *engineering* criterion of progress of Σ in agreement with the second law. This is very similar to what happens in *elastoplasticity* for the progress of the plasticized zone, or in *fracture* with the progress of the crack tip curve, or still in damage with the progress of a damaged zone. As a matter of fact, all these phenomena of irreversible progress belong in the same picture of driving forces acting on defects following the inclusive view of *Eshelbian mechanics* on the material manifold. The ultimate result will reside in a more or less regular relationship

$$\overline{V}_N=\mathcal{V}(f_\Sigma;\overline{\theta}) \tag{46}$$

with

$$\mathcal{V}(f_\Sigma;\overline{\theta})f_\Sigma\geq 0\,. \tag{47}$$

Contrary to other mentioned cases, whose progress may only be in one direction, $\overline{V}_N$ may have here a positive or negative sign depending on the sign of $\mathcal{HUGO}$.[1]

[1] Literary minded readers will have noticed that $\overline{V}_N$, essentially the component of a 'vector', is the thermodynamical conjugate of $\mathcal{HUGO}$. We feel 'miserable'.

5. Examples of Applications

The range of applications of the formulation of Sections 3 and 4 is extremely large by virtue of the very generality of expression (9). In particular, α may be either an *observable* or an *internal* variable of state. The simplest example is that of the classical theory of *elastic conductors of heat* for which we simply ignore α in (9) or, rather, one of the α's is none other than θ but $\nabla_R\theta$ is not present in W – see References [15, 16] for phase-transition fronts and DASCALU and MAUGIN [18] for applications to thermoelastic fracture. This applies to the case of shape-memory alloys. Another straightforward application is the quasi-electrostatics of linear or *non-linear electrelastic bodies* (e.g., piezoelectric ceramics). In this case α is taken as the electrostatic potential $\Phi(\mathbf{X}, t)$ in the material description so that the material electric field is given by $\mathbf{E} = -\nabla_R\Phi$. *Gauge invariance* requires that W does not depend explicitly on Φ, so that we are left with $W = W(\mathbf{F}, \mathbf{E} = -\nabla_R\Phi, \theta; \mathbf{X})$. With $\Pi = -\partial\overline{W}/\partial E$, the material electric polarization, there is no real local electric dissipation at regular material points unless we account for some *electric relaxation* (see MAUGIN et al. [19] for the formulation of this phenomenon). The formulation that follows from Section 3 is entirely compatible with that obtained in a different manner in the study of electroelastic fracture (DASCALU and MAUGIN [20]). As to the case of phase-transition fronts (Sect. 4), on account of the natural electric jump condition $\mathbf{N}\cdot[\mathbf{D}] = 0$, where $\mathbf{D}$ is the material electric displacement, and of the 'gauge' condition $[\partial\Phi/\partial t|\mathbf{x}] = 0$ at Σ, an expression of the type (44) follows for the power of the scalar driving force with the thermoelectroelastic Hugo functional given by (cf., MAUGIN and TRIMARCO [21])

$$\mathcal{HUGO} = [\overline{W}(\mathbf{F}, \mathbf{E}, \theta) - \mathbf{N}\cdot(\langle\mathbf{T}\rangle\cdot\mathbf{F} - \langle\mathbf{D}\rangle\otimes\mathbf{E})\cdot\mathbf{N}]\,, \tag{48}$$

where $\tilde{W}$ is the electric enthalpy and $\mathbf{D} = -\partial\tilde{W}/\partial\mathbf{E} = J_F\mathbf{C}^{-1}\cdot\mathbf{E} + \Pi$.

One could also consider the cases where α is a damage parameter, a degree of moisture, a scalar quasistatic magnetic potential, the wave function of superconducting pairs (in elastic superconductors [22]), or a rotational internal degree of freedom such as in *polar* elastic crystals or in liquid crystals (see Ref. [6] for a discussion of these cases). But the case of *elastic ferromagnets* is particularly enlightening because of peculiar features of magnetic-spin inertia.

In elastic ferromagnets, taking one of the α's as the scalar magnetic potential Φ, such that the material magnetic field be given by $\mathbf{H} = -\nabla_R\Phi$, and the other α as the actual magnetization per unit mass $\mu(\mathbf{X}, t)$, and applying a gauge condition to get rid of the explicit dependence upon Φ, we consider (9) in the following form:

$$W = \overline{W}(\mathbf{F}, \mathbf{H}, \mu, \nabla_R\mu, \theta; \mathbf{X})\,. \tag{49}$$

Detailed elements of the now classical theory of finitely deformable ferromagnets may be found in Chapter 6 of MAUGIN [23]. Applying the formalism of *Eq.* (13) it can be shown that the right-hand side gives rise to a *ferromagnetic material force* (compare to FOMETHE and MAUGIN [24]):

$$\mathbf{f}^{ferro} = \rho_0 \mathbf{B}^{eff} \cdot (\nabla_R \mu)^T , \tag{50}$$

where $\mathbf{B}^{eff}$ is the *effective* magnetic induction which defines the Larmor frequency of the magnetic-spin precession by

$$\omega = -\gamma \mathbf{B}^{eff}, \quad \mathbf{B}^{eff} = -\rho_0^{-1} \delta W / \delta \mu, \tag{51}$$

where γ is the gyromagnetic ratio. With

$$\partial \mu / \partial t|_X - \omega \times \mu = 0 , \tag{52}$$

the spin-precession equation respecting the saturation condition $|\mu| = \text{const.}$, it is shown that the ferromagnetic material force (50) is none other than a *Dalembertian* way of taking the magnetic-spin inertia into account as the latter, in a classical continuum formulation, cannot be integrated into a definite kinetic energy (gyroscopic feature) or $\mathbf{f}^{ferro}$ written as the time derivative of a canonical ferromagnetic momentum $\mathcal{P}^{ferro}$. Simultaneously, if we do not envisage spin-lattice relaxation, (14) transforms into a strict conservation law and (26) holds without any intrinsic entropy source. The formalism of Section 4 above holds good, in particular *Eq.* (43), and thus (44), because of the spin-boundary condition and of the coherency condition applied to the spin dynamics. We refer the reader to FOMETHE and MAUGIN [25] for details and further developments along these lines. We expect that the above-given examples, although sketchy, demonstrate the power and wide applicability of the presented formulation.

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