PERIODICA POLYTECHNICA SER. CHEM. ENG. VOL. 42, NO. 1, PP. 5-14 (1998)

# THERMODYNAMICS OF PLASTIC BODIES

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Received: March 30, 1997

## Abstract

The paper consists of three parts. The first contains well-known equations and relations. The second part quotes papers [3] and [5]. The third part raises the investigation of the role of the wave propagation in the case of the constitutive equation [1]. This paper generalizes the results of [1]. A new, so far unknown function is applied. In the case where the mechanical and thermodynamic waves can be independent, the conditions of the unknown functions are determined.

*Keywords:* elastic-plastic solid, thermo-mechanical wave, speed of wave.

## 1. Mechanical and Thermodynamic Equations and Relations

Small deformations of solids will be investigated in the following section and Cartesian coordinates will be used.

The mechanical equations are the equation of motion

$$\sigma^{ij}_{;j} + q^i = \rho \dot{v}^i$$

and (1)

$$\sigma^{ij} = \sigma^{ji} \, . \tag{2}$$

The kinematic equation is

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \qquad \text{or} \qquad \dot{\varepsilon}_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right) \, , \tag{3}$$

where the notations are: $\sigma^{ij}$ stress, $\varepsilon_{ij}$ strain, $\dot{\varepsilon}_{ij}$ strain rate, $q_i$ volume force, $v^i$ velocity of the particle and $\rho$ density of mass.

[‡]This work was supported by the Hungarian-American Fund for Scientific Cooperation. The second author would like to thank also the OTKA for support under contract F 017331.

The thermodynamical equations and relations are as follows. First

$$\rho\dot{u} = \sigma^{ij}\dot{\varepsilon}_{ij} - h^i_{,i} \tag{4}$$

is the first law of thermodynamics.

By using Helmholz's function $H = u - Ts$

$$\rho\dot{H} + \rho\dot{T}s + \rho T\dot{s} = \sigma^{ij}\dot{\varepsilon}_{ij} - h^i_{,i}\,. \tag{5}$$

We assume that the free energy for an elastic body is [2, 4]

$$H = \frac{1}{2}C^{pqmn}\varepsilon_{pq}\varepsilon_{mn} + B^{pq}\varepsilon_{pq}(T - T_0) + \frac{1}{2}\alpha(T - T_0)^2. \tag{6}$$

We obtain Duhamel–Neuman's law from (6) because

$$\begin{aligned} \sigma^{kl} &= \rho\frac{\partial H}{\partial\varepsilon_{kl}}\,, \\ \sigma^{kl} &= \rho[C^{klmn}\varepsilon_{mn} + B^{kl}(T - T_0)]\,, \end{aligned} \tag{7}$$

where $T$ denotes temperature, $s$ specific entropy and $h^i$ heat flux vector.

The second law of thermodynamics is

$$-\rho\left(\dot{H} + \dot{T}s\right) + \sigma^{kl}\dot{\varepsilon}_{kl} - \frac{1}{T}h^k T_{,k} \geq 0 \tag{8}$$

or Clausius–Duhem's inequality

$$\sigma^{kl}\dot{\varepsilon}_{kl} - \rho\left(\dot{H} + \dot{T}s\right) \geq 0\,. \tag{9}$$

The other form of the second law of thermodynamics is [4]

$$\rho T\dot{s} + h^i_{,i} = \sigma_s \tag{10}$$

and

$$h^i T_{,i} \leq 0 \qquad \text{and} \qquad \sigma_s > 0\,. \tag{11}$$

The Vernotte–Cattaneo's equation [5]

$$\tau\dot{h}_i = -\alpha_i^j T_{,j} - h_i\,. \tag{12}$$

## 2. The Elastic–Plastic Solid

We introduce a tensor $\varepsilon''_{kl}$ — having the same invariance properties as $\varepsilon_{kl}$ — which we shall call a 'plastic strain' tensor, and we postulate a constitutive equation for $\varepsilon_{kl}$ in the form

$$\varepsilon_{kl} = \varepsilon_{kl}(\sigma^{kq}, T, \varepsilon''_{pq}). \tag{13}$$

If we denote the difference of the two tensors $\varepsilon_{kl}$ and $\varepsilon''_{kl}$ by $\varepsilon'_{kl}$ so that

$$\varepsilon'_{kl} = \varepsilon_{kl} - \varepsilon''_{kl}, \tag{14}$$

then it follows from (13) that the constitutive equation for $\varepsilon'_{kl}$ must have the form

$$\varepsilon'_{kl} = \varepsilon'_{kl}(\sigma^{pq}, T, \varepsilon''_{pq}). \tag{15}$$

The tensors $\varepsilon'_{kl}$ and $\varepsilon''_{kl}$ cannot be expressed in terms of the displacements by formulae such as $(3)_2$, but only their sum $\varepsilon_{kl}$ is given by (3). We shall call $\varepsilon'_{kl}$ an 'elastic strain' tensor. Equation (15) has a unique inverse of form [3, 5]

$$\sigma^{kl} = (\varepsilon'_{pq}, \varepsilon''_{pq}, T)\ . \tag{16}$$

We introduce moreover a constitutive equation for the plastic strain $\varepsilon''_{kl}$. Consider the equation

$$f(\sigma^{kl}, \varepsilon''_{kl}, T) = \kappa, \tag{17}$$

where $f$ is a continuously differentiable function of its variables and $\kappa$ is a scalar which depends in some way on the whole history of motion.

We postulate a constitutive equation for $\dot{\varepsilon}''_{kl}$ in the form

$$\dot{\varepsilon}''_{kl} = g_{kl}(\sigma^{pq}, \dot{\sigma}^{pq}, \varepsilon''_{pq}; T, \dot{T}), \quad f = \kappa\ . \tag{18}$$

We further add (18) that

$$\dot{\varepsilon}''_{kl} = 0 \qquad \text{when} \qquad \kappa = \kappa^* \qquad \text{and} \qquad \dot{\kappa} = 0\ . \tag{19}$$

For a given value of $\kappa$ and $\varepsilon''_{kl}$, *Eq.* (17) represents a surface in seven-dimensional space (six components of stress and one component of temperature). Thus, we say that a point lies on the surface (17) if

$$\frac{\partial f}{\partial \sigma^{kl}}\dot{\sigma}^{kl} + \frac{\partial f}{\partial T}\dot{T} = 0, \qquad f = \kappa^* \tag{20}$$

since $\dot{\varepsilon}''_{kl}$ vanishes on the surface. Similarly, a point in stress and temperature space is inside or outside the surface (17) according to

$$\frac{\partial f}{\partial \sigma^{kl}}\dot{\sigma}^{kl} + \frac{\partial f}{\partial T}\dot{T} \le 0 \quad \text{or} \quad \ge 0 \qquad \text{on} \qquad f = \kappa^*\ . \tag{21}$$

We call a state neutral when

$$\frac{\partial f}{\partial \sigma^{kl}}\dot{\sigma}^{kl} + \frac{\partial f}{\partial T}\dot{T} = 0, \qquad f = \kappa, \qquad \dot{\kappa} = 0$$

or unloading when

$$\dot{\varepsilon}''_{kl} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial \sigma^{kl}}\dot{\sigma}^{kl} + \frac{\partial f}{\partial T}\dot{T} \leq 0 \quad \text{on} \quad f = \kappa, \quad \dot{\kappa} = 0$$

or when $f \leq 0$, $\dot{\kappa} = 0$ and we call loading when

$$\frac{\partial f}{\partial \sigma^{kl}}\dot{\sigma}^{kl} + \frac{\partial f}{\partial T}\dot{T} \geq 0, \quad f = \kappa \qquad \text{and} \qquad \dot{\kappa} \neq 0.$$

[3, 5]. The surfaces (17) are usually yield surfaces or loading surfaces.

From (17) and (18),

$$\dot{\kappa} = \dot{\kappa}(\sigma^{kl}, \varepsilon''_{kl}, T, \dot{\sigma}^{kl}, \dot{\varepsilon}''_{kl}, \dot{T}) . \tag{22}$$

We restrict further discussions to the case when $\dot{\kappa}$ is a linear function of $\dot{\sigma}^{kl}$, $\dot{\varepsilon}''_{kl}$ and $\dot{T}$ and $g_{kl}$ is a linear function of $\dot{\sigma}_{kl}$ and $\dot{T}$. In view of $\dot{\kappa} = 0$ when $\dot{\varepsilon}''_{kl} = 0$ which must be statisfied for all $\dot{\sigma}^{kl}$ and $\dot{T}$, we assume that

$$\dot{\kappa} = h^{kl}(\sigma^{pq}, \varepsilon''_{pq}, T)\dot{\varepsilon}''_{kl} \tag{23}$$

and also write

$$\dot{\varepsilon}''_{kl} = g_{kl} = \alpha_{klmn}\dot{\sigma}^{kl} + \alpha_{kl}\dot{T}, \tag{24}$$

where $h^{kl}$, $\alpha_{klmn}$ and $\alpha_{kl}$ are tensor functions of $\sigma^{pq}$, $\varepsilon''_{pq}$ and $T$. If $\dot{\varepsilon}''_{kl}$ and $\dot{\kappa}$ are equal to zero, from (24) and (20) we obtain

$$\alpha_{klmn}\dot{\sigma}^{mn} + \alpha_{kl}\dot{T} = 0$$

and

$$\frac{\partial f}{\partial \sigma^{mn}}\dot{\sigma}^{mn} + \frac{\partial f}{\partial T}\dot{T} = 0.$$

It results from these equations that

$$\alpha_{kl} = \lambda\beta_{kl}\frac{\partial f}{\partial T} \qquad \text{and} \qquad \alpha_{klmn} = \lambda\beta_{kl}\frac{\partial f}{\partial \sigma^{mn}}, \qquad \lambda \geq 0 . \tag{25}$$

(24) and (25) implies [3, 5]

$$\dot{\varepsilon}''_{kl} = \Lambda\beta_{kl} , \tag{26}$$

where

$$\Lambda = \lambda\left(\frac{\partial f}{\partial \sigma_{mn}}\dot{\sigma}_{mn} + \frac{\partial f}{\partial T}\dot{T}\right) . \tag{27}$$

Now we return to the thermodynamical laws (5) and (6). The variable $v_{kl}$ in them is equal to $\dot{\varepsilon}_{kl}$ because the deformation is small.

We assume that the free energy, entropy and thermal flux are

$$\begin{aligned} H &= H(\varepsilon'_{kl}, \varepsilon''_{kl}, T)\,, \\ s &= s(\varepsilon'_{kl}, \varepsilon''_{kl}, T)\,, \end{aligned}$$

and

$$h^k = h^k(T, T_{,m}, \varepsilon_{pq}, \varepsilon''_{pq})\,.$$

With these functions, *Eqs.* (5) and (6) may be written as

$$\rho r - \rho\left(s + \frac{\partial H}{\partial T}\right)\dot{T} + \left(\sigma^{kl} - \rho\frac{\partial H}{\partial \varepsilon'_{kl}}\right)\dot{\varepsilon}'_{kl} + \left(\sigma^{kl} - \rho\frac{\partial H}{\partial \varepsilon''_{kl}}\right)\dot{\varepsilon}''_{kl} - \rho\dot{s}T - h^k{}_{;l} = 0 \tag{28}$$

and

$$-\rho\left(s + \frac{\partial H}{\partial T}\right)\dot{T} + \rho\left(\sigma^{kl} - \frac{\partial H}{\partial \varepsilon'_{kl}}\right)\dot{\varepsilon}'_{kl} + \left(\sigma^{kl} - \rho\frac{\partial H}{\partial \varepsilon''_{kl}}\right)\dot{\varepsilon}''_{kl} - \frac{h^k T_{,k}}{T} \geq 0 \tag{29}$$

both of which must hold during loading, as well as unloading. In particular, if we consider unloading during which $\dot{\varepsilon}''_{kl} = 0$ and $f \leq \kappa$, then the stress and temperature in any point within the loading surface and for all arbitrary and independent $\dot{T}$ and $\dot{\varepsilon}''_{kl}$ follow the relations

$$s = -\frac{\partial H}{\partial T}\,, \tag{30}$$

$$\sigma^{kl} = \rho\frac{\partial H}{\partial \varepsilon'_{kl}}\,. \tag{31}$$

Using (30) and (31), *Eqs.* (28) and (29) now become

$$\rho r + \left(\sigma^{kl} - \rho\frac{\partial H}{\partial \varepsilon''_{kl}}\right)\varepsilon''_{kl} - \rho\dot{s}T - h^k{}_{;k} = 0 \tag{32}$$

and

$$\left(\sigma^{kl} - \rho\frac{\partial H}{\partial \varepsilon''_{kl}}\right)\varepsilon''_{kl} - \frac{h^k T_{,k}}{T} \geq 0\,, \tag{33}$$

respectively.

By combining (33), (26) and (27) we have during loading [3, 5]

$$\lambda\beta_{kl}\left(\sigma^{kl} - \rho\frac{\partial H}{\partial \varepsilon''_{kl}}\right)\left(\frac{\partial f}{\partial \sigma^{kl}}\dot{\sigma}^{kl} + \frac{\partial f}{\partial T}\dot{T}\right) - \frac{h^k T_{;k}}{T} \geq 0\,, \tag{34}$$

whenever

$$\lambda\left(\frac{\partial f}{\partial \sigma^{kl}}\dot{\sigma}^{kl} + \frac{\partial f}{T}\dot{T}\right) \geq 0, \qquad f = \kappa, \qquad \lambda \geq 0\,. \tag{35}$$

The inequality (33) or equivalently (34) holds only during loading, whereas during neutral loading and unloading (since $\dot{\varepsilon}''_{kl} = 0$) we have

$$-h^k T_{;k} \geq 0 . \tag{36}$$

Also, the energy equation (32) holds during loading but during neutral loading or unloading, this is reduced to

$$\rho r - \rho \dot{s} T - h^k{}_{;k} = 0 . \tag{37}$$

We assume for simplicity that

$$H = H'(\varepsilon'_{kl}, T) + H''(\varepsilon''_{kl}, T) . \tag{38}$$

Now (31) is

$$\sigma^{kl} = \rho \frac{\partial H'}{\partial \varepsilon'_{kl}} . \tag{39}$$

When $H'$ is similar to *Eq.* (6) then we obtain *Eq.* (7) but now $\varepsilon'_{kl}$ stands instead of $\varepsilon_{kl}$. This equation has the following inverse form

$$\varepsilon'_{kl} = K_{klmn}\sigma^{mn} - L_{kl}(T - T_0). \tag{40}$$

We can write (26) and (27) in the form

$$\dot{\varepsilon}''_{kl} = D_{klmn}\sigma^{mn} + A_{kl}\dot{T} . \tag{41}$$

Combining (40) and (41) we obtain

$$\dot{\varepsilon}_{kl} = K_{klmn}\dot{\sigma}^{mn} + (\dot{K}_{klmn} + D_{klmn})\sigma^{mn} + (A_{kl} - L_{kl})\dot{T} - \dot{L}_{kl}(T - T_0) . \tag{42}$$

*Eq.* (42) can be accepted as constitutive equation for thermoplasticity, during loading state.

## 3. Second Order Thermo-Mechanical Wave

The basic functions are called stress $\sigma^{ij}$, strain $\varepsilon_{ij}$, velocity $v^i$, temperature $T$, internal specific energy $u$, heat flux $h^i$, and specific entropy $s$.

We will investigate the case in which the basic functions are continuous but their derivatives have a jump along the wave surface $\varphi(x^i) = 0$. For example, $A$ is an arbitrary function, which is continuous along the wave surface $[A] = 0$ ( [] denotes the jump) and their derivatives are not continuous $[A_{,\hat{p}}] = \alpha\varphi_{,\hat{p}} \equiv \alpha\varphi_{\hat{p}}$. Here $\varphi = \varphi(x_{\hat{p}})$ is the wave surface, where $\hat{p} = 1, 2, 3, 4$ and $x_4 \equiv t$, that is, $x_4$ denotes time.

The normal unit vector of the wave front is $n_k = \frac{\varphi_{,k}}{\sqrt{\delta^{lm}\varphi_{,l}\varphi_{,m}}}$ and the wave propagation velocity $c = -\frac{\frac{\partial\varphi}{\partial t}}{\sqrt{\delta^{lm}\varphi_{,l}\varphi_{,m}}}$, where $\frac{\partial\varphi}{\partial t} \equiv \varphi_{,4}$ and $k,l,m = 1,2,3$. The functions $A$ are $\sigma^{ij}$, $v^i$, $\varepsilon_{ij}$, $T$, $u$, $h^i$, $s$ $\mu^{ij}$, $v^i$, $\kappa_{ij}$, $\vartheta$, $\lambda$, $\chi^i$, $\sigma$ have jumps [1].

In the case of the second order thermomechanical wave the compatibility equations are

$$\mu^{ij} n_j = -\rho v^i c \,, \tag{43}$$

$$2\kappa_{ij} c = -(v_i n_j + v_j n_i) \,, \tag{44}$$

$$\kappa_{kl} = K_{klmn}\mu^{mn} + \left(\frac{\partial K_{klmn}}{\partial \varepsilon_{pq}}\kappa_{pq} + \frac{\partial K_{klmn}}{\partial T}\vartheta\right)\sigma^{mn} + (A_{kl} - L_{kl})\vartheta -$$

$$-\left(\frac{\partial L_{kl}}{\partial \varepsilon_{pq}}\kappa_{pq} + \frac{\partial L_{kl}}{\partial T}\vartheta\right)(T - T_0) \,,$$

where

$$K_{klmn}(\varepsilon_{pq}, T)$$

and

$$L_{kl}(\varepsilon_{pq}, T).$$

This equation can be written in the form

$$\overline{A}_{kl}{}^{pq}\kappa_{pq} = K_{klmn}\mu^{mn} + B_{kl}\vartheta$$

or

$$\mu^{rs} = \overline{\overline{A}}^{rspq}\kappa_{pq} - \overline{B}^{rs}\vartheta \,. \tag{45}$$

The other compatibility equations are

$$c\rho\lambda = \sigma^{ij}\kappa_{ij}c + \chi^i n_i \tag{46}$$

$$c\left(\rho T\sigma - \frac{\partial\sigma_s}{\partial\dot\varepsilon_{kl}}\kappa_{kl} - \frac{\partial\sigma_s}{\partial\dot T}\vartheta\right) = n_i\left(\chi^i - \frac{\partial\sigma_s}{\partial T_{,i}}\right), \tag{47}$$

when $\sigma_s(\dot\varepsilon_{ij}, \dot T, T_{;i}, \varepsilon_{ij}, T) > 0$.

$$c\tau\chi^i = \alpha^{ij} n_j \vartheta \,. \tag{48}$$

*Eqs.* (43)–(48) contain twenty-one unknown functions. The number of the equations is twenty. One more equation is necessary.

Let the missing one equation be $g = 0$, where

$$g = g(\dot\sigma_{kl}, \dot\varepsilon_{kl}, \dot h^i, h^i{}_{,j}, \dot T, T_{,j}, \sigma_{kl}, \varepsilon_{kl}, h^i, T) \,.$$

The compatibility equation for $g$ is

$$n_j\left(\frac{\partial g}{\partial T_{,j}}\vartheta+\frac{\partial g}{\partial h^i{}_{,j}}\chi^i\right)=c\left(\frac{\partial g}{\partial\dot\varepsilon_{ik}}\kappa_{ik}+\frac{\partial g}{\partial\dot\sigma_{kl}}\mu^{kl}+\frac{\partial g}{\partial\dot T}\vartheta+\frac{\partial g}{\partial\dot h^i}\chi^i\right). \tag{49}$$

Using (45) and (48) *Eqs.* (43),. . . ,(49) can be written as

$$n_s\left(\overline{\overline{A}}^{rspq}\kappa_{pq}-\overline{B}^{rs}\vartheta\right)+\rho c v_r=0\,, \tag{50}$$

$$\kappa_{pq}=-\frac{1}{2c}\left(v_p n_q+v_q n_p\right)\,, \tag{51}$$

$$c\rho\lambda-\sigma^{ij}c\kappa_{ij}-\frac{1}{\tau c}\alpha^{ij}n_i n_j\vartheta=0\,, \tag{52}$$

$$c\left(\rho T\sigma-\frac{\partial\sigma_s}{\partial\dot\varepsilon_{kl}}\kappa^{kl}\right)-\left[n_i\left(\frac{1}{\tau c}\alpha^{ij}n_j-\frac{\partial\sigma_s}{\partial T_{,i}}\right)+c\frac{\partial\sigma_s}{\partial\dot T}\right]\vartheta=0\,, \tag{53}$$

$$\left[n_j\left(\frac{\partial g}{\partial T_{,j}}+\frac{\partial g}{\partial h^i{}_{,j}}\frac{1}{\tau c}\alpha^{ik}n_k\right)-c\left(\frac{\partial g}{\partial\dot T}+\frac{\partial g}{\partial\dot h^i}\alpha^{ij}n_j\frac{1}{\tau c}\right)+c\overline{B}^{kl}\frac{\partial g}{\partial\dot\sigma^{kl}}\right]\vartheta-$$

$$-c\left(\frac{\partial g}{\partial\dot\varepsilon_{pq}}+\frac{\partial g}{\partial\dot\sigma^{kl}}\overline{\overline{A}}^{klpq}\right)\kappa_{pq}=0\,, \tag{54}$$

where $\sigma_s\geq 0$.

By substituting (51) a $6^{\text{th}}$ order linear homogeneous equation can be obtained for $v_1$, $v_2$, $v_3$, $\lambda$, $\vartheta$, $\sigma$.

$$\left[-n_j\overline{\overline{A}}^{ijkl}\left(n_l\delta^s_k+\delta^s_l n_k\right)+2\rho\;c^2\delta^{si}\right]v_s-2cn_j\overline{B}^{ij}\vartheta=0\,, \tag{55}$$

$$\frac{1}{2}\tau c\sigma^{ij}\left(\delta^s_i n_j+\delta^s_j n_i\right)v_s+\tau c^2\lambda-\alpha^{ij}n_i n_j\vartheta=0\,, \tag{56}$$

$$\frac{1}{2}\tau c\frac{\partial\sigma_s}{\partial\dot\varepsilon_{kl}}\left(\delta^s_k n_l+\delta^s_l n_k\right)v_s+\rho\tau c^2 T\sigma-$$

$$-\left[n_i\left(\alpha^{ij}n_j-\tau c\frac{\partial\sigma_s}{\partial T_{,i}}\right)+\tau c^2\frac{\partial\sigma_s}{\partial\dot T}\right]\vartheta=0\,, \tag{57}$$

$$\frac{1}{2}\tau c\left(\frac{\partial g}{\partial\dot\varepsilon_{pq}}+\overline{\overline{A}}^{klpq}\frac{\partial g}{\partial\dot\sigma^{kl}}\right)\left(\delta^s_p n_q+\delta^s_q n_p\right)v_s+\left[n_j\left(\tau c\frac{\partial g}{\partial T_{,j}}+\frac{\partial g}{\partial h^i{}_{,j}}\alpha^{ik}n_k\right)-\right.$$

$$\left.-\left(\tau c^2\frac{\partial g}{\partial\dot T}+\frac{\partial g}{\partial\dot h^i}\alpha^{ij}n_j c\right)+\tau c^2\overline{B}^{kl}\frac{\partial g}{\partial\dot\sigma^{kl}}\right]\vartheta=0\,. \tag{58}$$

The determinant of it is zero. This determinant is a $12^{\text{th}}$ order algebraic equation. We require 4 real roots.

From this we can study the behavior of functions $\sigma_s > 0$, $g$.
When

$$\frac{\partial g}{\partial \dot{\varepsilon}_{pq}} + \overset{=}{A}^{kipq} \frac{\partial g}{\partial \dot{\sigma}^{kl}} = 0 , \tag{59}$$

then the determinant of (55), (56), (57) and (58) is the simplest form. In this case it is a product of two determinants and the mechanical wave is independent of the thermodynamical wave. That is

$$\begin{vmatrix} 2\rho c^2 + M^{11} & M^{12} & M^{13} \\ M^{21} & 2\rho c^2 + M^{22} & M^{23} \\ M^{31} & M^{32} & 2\rho c^2 + M^{33} \end{vmatrix} \cdot$$

$$\cdot \begin{vmatrix} -\alpha^{ij} n_i n_j & \tau c^2 & 0 \\ -\alpha^{ij} n_i n_j + \tau n_i \frac{\partial \sigma_s}{\partial T_{,i}} c + \tau \frac{\partial \sigma_s}{\partial \dot{T}} c^2 & 0 & \rho \tau T c^2 \\ n_j \frac{\partial g}{\partial h^i_{\ ,j}} \alpha^{ij} n_k + n_j \left( \tau \frac{\partial g}{\partial T_{,j}} - \frac{\partial g}{\partial \dot{h}_i} \alpha^{ij} \right) c + \tau \left( \overline{B}^{kl} \frac{\partial g}{\partial \dot{\sigma}^{kl}} - \frac{\partial g}{\partial \dot{T}} \right) c^2 & 0 & 0 \end{vmatrix} = $$

$$= 0$$

it is satisfied when the first determinant is equal to zero or the second one is equal to zero. From the first equation the speed of propagation of the mechanical wave is obtained, and from the second one the speed of propagation of a thermodynamical wave is obtained.

The first equation has the following form

$$\alpha c^6 + \beta c^4 + \gamma c^2 + \delta = 0 .$$

The second is

$$\rho \tau^2 T c^4 \left[ n_j \frac{\partial g}{\partial h^i_{\ ,j}} \alpha^{ik} n_k + \right.$$

$$\left. + n_j \left( \tau \frac{\partial g}{\partial T_{,j}} - \frac{\partial g}{\partial \dot{h}_i} \alpha^{ij} \right) c + \tau \left( \overline{B}^{kl} \frac{\partial g}{\partial \dot{\sigma}^{kl}} - \frac{\partial g}{\partial \dot{T}} \right) c^2 \right] = 0 . \tag{60}$$

$c = 0$ is a multiplicity four root of (60). *Eq.* (60) has got one positive and one negative real root. The condition of this is

$$\left( n_j \frac{\partial g}{\partial h^i_{\ ,j}} \alpha^{ik} n_k \right) \left[ \tau \left( \overline{B}^{kl} \frac{\partial g}{\partial \dot{\sigma}^{kl}} - \frac{\partial g}{\partial \dot{T}} \right) \right] < 0. \tag{61}$$

When the mechanical and thermodynamical waves are coupled, for example, when the unknown function depends on the rate of specific entropy $\dot{s}$, the equations of wave propagation are more complicated.

Finally, stress is divided into two parts, that is $\sigma^{ij} = {}_E\, \sigma^{ij} + {}_D\, \sigma^{ij}$. The first is the elastic stress (31) the second is the plastic stress, that is, ${}_D\sigma^{ij} = \frac{\partial \sigma_s}{\partial \dot{\varepsilon}_{ij}}$ with the condition that $\frac{\partial \sigma_s}{\partial \dot{\varepsilon}_{ij}} > 0$.

The stress is $\sigma^{ij} = \frac{\partial H}{\partial \varepsilon_{ij}} + \frac{\partial \sigma_s}{\partial \dot{\varepsilon}_{ij}}$.

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